maketitle ?

Remark. It remains only to prove the properties 5.4, 5.5, 5.6 of the middles p.35. I already proved the lemmas 4.9 p.28 and 4.11 p.29 on my notes, I will recopy soon. The proofs from the beginning to the lemma 3.3 p.11 included have been successfully implemented in Coq, except for the three technical lemmas 2.5, 2.6, 2.7.

1 FIELDS

For the Coq implementation, the fields are computed using booleans but the results will be proven using propositions as recommanded in Software Foundations (Benjamin Pierce). The definition are simplified (quantifier elimination) for the implementation, as opposed to the paper (cite finitization).

The problem is one dimensional, where each cell is a number between 1 and n.

Axiom 1.1 (At least three cells)**.**

$$n \geq 3$$

The boolean function gen is given by the evolution.

Axiom 1.2 (At least one cell is a general)**.**

$$\exists c, 1 \leq c \leq n \wedge \mathrm{gen}(c) = \mathrm{true}$$

In the following definitions, we assume that or , and and if $\ldots$ then $\ldots$ else $\ldots$ are the standard booleans operations.

Definition 1.3 (Input Field)**.**

$$\begin{aligned}
\mathrm{inp}_0(c) &\stackrel{\text{def}}{=} \mathrm{gen}(c) \\
\mathrm{inp}_{t+1}(c) &\stackrel{\text{def}}{=} \mathrm{inp}_t(c-1) \text{ or } \mathrm{inp}_t(c) \text{ or } \mathrm{inp}_t(c+1)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Inp}_0(c) &\stackrel{\text{def}}{=} \mathrm{gen}(c) = \mathrm{true} \\
\mathrm{Inp}_{t+1}(c) &\stackrel{\text{def}}{=} \mathrm{Inp}_t(c-1) \vee \mathrm{Inp}_t(c) \vee \mathrm{Inp}_t(c+1) \qquad (1)
\end{aligned}$$

Like this definition, the boolean fields will be written with lowercase, and the proposition fields in uppercase. The Coq file contains the proof of equivalence, but they will be admitted in this report.

Lemma 1.4 (Equivalence for Inp)**.**

$$\forall tc, \mathrm{Inp}_t(c) \Leftrightarrow \mathrm{inp}_t(c) = \mathrm{true}$$

We assume in the following that the cells are labeled from 1 to n. For the sake of clarity, the $=$ and $<$ will not be distinguished from their boolean equivalent, as it is in Coq. The recursive definition of the proposition fields is:

Definition 1.5 (Proposition Fields)**.**

$$\begin{aligned}
\mathrm{Brd}_t^0(c) &\stackrel{\text{def}}{=} \mathrm{Inp}_t(c) \wedge (1 = c \vee c = n) \\
\mathrm{Brd}_t^{\ell+1}(c) &\stackrel{\text{def}}{=} \mathrm{Brd}_t^{\ell}(c) \vee \mathrm{Mid}_t^{\ell}(c)
\end{aligned} \tag{2}$$

$$\begin{aligned}
\mathrm{Ins}_t^0(c) &\stackrel{\text{def}}{=} \mathrm{Inp}_t(c) \wedge 1 < c \wedge c < n \\
\mathrm{Ins}_0^{\ell+1}(c) &\stackrel{\text{def}}{=} \mathrm{False} \\
\mathrm{Ins}_{t+1}^{\ell+1}(c) &\stackrel{\text{def}}{=} \mathrm{Ins}_{t+1}^{\ell}(c) \wedge \mathrm{Sta}_{t+1}^{\ell}(c) \\
&\quad \wedge \Big(\mathrm{dst}_{t+1}^{\ell}(c) < \mathrm{dst}_t^{\ell}(c-1) \vee \mathrm{dst}_{t+1}^{\ell}(c) < \mathrm{dst}_t^{\ell}(c-1)\Big)
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mathrm{Sta}_0^{\ell}(c) &\stackrel{\text{def}}{=} \mathrm{Brd}_0^{\ell}(c) \\
\mathrm{Sta}_{t+1}^{\ell}(c) &\stackrel{\text{def}}{=} \mathrm{Brd}_{t+1}^{\ell}(c) \\
&\quad \vee \Big(\mathrm{dst}_{t+1}^{\ell}(c) = 1 + \mathrm{dst}_t^{\ell}(c-1) \wedge \mathrm{Sta}_t^{\ell}(c-1)\Big) \\
&\quad \vee \Big(\mathrm{dst}_{t+1}^{\ell}(c) = 1 + \mathrm{dst}_t^{\ell}(c+1) \wedge \mathrm{Sta}_t^{\ell}(c+1)\Big)
\end{aligned} \tag{4}$$

$$\begin{aligned}
\mathrm{Mid}_0^{\ell}(c) &\stackrel{\text{def}}{=} \mathrm{False} \\
\mathrm{Mid}_{t+1}^{\ell}(c) &\stackrel{\text{def}}{=} \Bigg(\mathrm{dst}_{t+1}^{\ell}(c) > \max\Big(\mathrm{dst}_t^{\ell}(c-1), \mathrm{dst}_t^{\ell}(c+1)\Big) \\
&\qquad \wedge \mathrm{Sta}_t^{\ell}(c-1) \wedge \mathrm{Sta}_t^{\ell}(c+1)\Bigg) \\
&\quad \vee \Bigg(\mathrm{dst}_{t+1}^{\ell}(c) = \max\Big(\mathrm{dst}_t^{\ell}(c-1), \mathrm{dst}_t^{\ell}(c+1)\Big) \\
&\qquad \wedge \mathrm{Sta}_t^{\ell}(c-1) \wedge \mathrm{Sta}_t^{\ell}(c) \wedge \mathrm{Sta}_t^{\ell}(c+1)\Bigg)
\end{aligned} \tag{5}$$

where dst is an integer field computed along the booleans fields.

Coq cannot guess* how to compute such an intricated recursion, so the recursive definition of the booleans fields is sliced into abstract parts for different given levels. Firstly, using the input field, the border and inside fields are defined for the level 0:

Definition 1.6 (Border and Inside Fields at level 0)**.**

$$\mathrm{brd0}(t,c) \stackrel{\mathrm{def}}{=} \mathrm{inp}_t(c) \text{ and } (1 = c \text{ or } c = n)$$

$$\mathrm{ins0}(t,c) \stackrel{\mathrm{def}}{=} \mathrm{inp}_t(c) \text{ and } 1 < c \text{ and } c < n$$

Then, the distance, stability and middle fields are defined for every level ℓ, assuming that the border and inside fields are defined too at this level:

*At least at my knowledge...

Definition 1.7 (Distance, Stability and Middle Fields)**.**

$$\begin{aligned}
\mathrm{dstL}(0, c, \mathrm{insL}) &\stackrel{\text{def}}{=} 0\\
\mathrm{dstL}(t+1, c, \mathrm{insL}) &\stackrel{\text{def}}{=} \text{if } \mathrm{insL}(t+1, c)\\
&\quad\ \text{then } 1 + \min\Big(\mathrm{dstL}(t, c-1), \mathrm{dstL}(t, c+1)\Big)\\
&\quad\ \text{else } 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{staL}(0, c, \mathrm{brdL}, \mathrm{dstL}) &\stackrel{\text{def}}{=} \mathrm{brdL}(0, c)\\
\mathrm{staL}(t+1, c, \mathrm{brdL}, \mathrm{dstL}) &\stackrel{\text{def}}{=} \mathrm{brdL}(t+1, c)\\
&\quad\ \text{or } \Big(\mathrm{dstL}(t+1, c) = 1 + \mathrm{dstL}(t, c-1) \text{ and } \mathrm{staL}(t, c-1)\Big)\\
&\quad\ \text{or } \Big(\mathrm{dstL}(t+1, c) = 1 + \mathrm{dstL}(t, c+1) \text{ and } \mathrm{staL}(t, c+1)\Big)
\end{aligned}$$

$$\begin{aligned}
\mathrm{midL}(0, c, \mathrm{dstL}, \mathrm{staL}) &\stackrel{\text{def}}{=} \text{false}\\
\mathrm{midL}(t+1, c, \mathrm{dstL}, \mathrm{staL}) &\stackrel{\text{def}}{=} \Big(\mathrm{dstL}(t+1, c) > \max\Big(\mathrm{dstL}(t, c-1), \mathrm{dstL}(t, c+1)\Big)\\
&\qquad \text{and } \mathrm{staL}(t, c-1) \text{ and } \mathrm{staL}(t, c+1)\Big)\\
&\quad\ \text{or } \Big(\mathrm{dstL}(t+1, c) = \max\Big(\mathrm{dstL}(t, c-1), \mathrm{dstL}(t, c+1)\Big)\\
&\qquad \text{and } \mathrm{staL}(t, c-1) \text{ and } \mathrm{staL}(t, c) \text{ and } \mathrm{staL}(t, c+1)\Big)
\end{aligned}$$

Finally, the border and inside fields for the level $\ell + 1$ are defined using the fields defined for the level ℓ:

Definition 1.8 (Border and Inside Fields at level $\ell + 1$)**.**

$$\mathrm{brdS}(t, c, \mathrm{brdL}, \mathrm{midL}) \stackrel{\text{def}}{=} \mathrm{brdL}(t, c) \text{ or } \mathrm{midL}(t, c)$$

$$\begin{aligned}
\mathrm{insS}(0, c, \mathrm{insL}, \mathrm{dstL}, \mathrm{staL}) &\stackrel{\text{def}}{=} \text{false}\\
\mathrm{insS}(t+1, c, \mathrm{insL}, \mathrm{dstL}, \mathrm{staL}) &\stackrel{\text{def}}{=} \mathrm{insL}(t+1, c) \text{ and } \mathrm{staL}(t+1, c)\\
&\quad\ \text{and } \Big(\mathrm{dstL}(t+1, c) < \mathrm{dstL}(t, c-1) \text{ or } \mathrm{dstL}(t+1, c) < \mathrm{dstL}(t, c-1)\Big)
\end{aligned}$$

So the boolean fields should be defined by this mutual recursion:

$$\begin{aligned}
\mathrm{brd}_t^0(c) &\stackrel{\mathrm{def}}{=} \mathrm{brd0}(t, c)\\
\mathrm{brd}_t^{\ell+1}(c) &\stackrel{\mathrm{def}}{=} \mathrm{brdS}(t, c, \mathrm{brd}^\ell, \mathrm{mid}^\ell)
\end{aligned}$$

$$\begin{aligned}
\mathrm{ins}_t^0(c) &\stackrel{\mathrm{def}}{=} \mathrm{ins0}(t, c)\\
\mathrm{ins}_t^{\ell+1}(c) &\stackrel{\mathrm{def}}{=} \mathrm{insS}(t, c, \mathrm{ins}^\ell, \mathrm{dst}^\ell, \mathrm{sta}^\ell)
\end{aligned}$$

$$\mathrm{dst}_t^\ell(c) \stackrel{\mathrm{def}}{=} \mathrm{dstL}(t, c, \mathrm{ins}^\ell)$$

$$\mathrm{sta}_t^\ell(c) \stackrel{\mathrm{def}}{=} \mathrm{staL}(t, c, \mathrm{brd}^\ell, \mathrm{dst}^\ell)$$

$$\mathrm{mid}_t^\ell(c) \stackrel{\mathrm{def}}{=} \mathrm{midL}(t, c, \mathrm{dst}^\ell, \mathrm{sta}^\ell)$$

But Coq cannot guess the decreasing argument. So, instead, we substitute the schemata to obtain only one mutual recursion for brd and ins, and thereafter define the other fields:

Definition 1.9 (Boolean Fields)**.**

$$\begin{aligned}
\mathrm{brd}^0 &\stackrel{\mathrm{def}}{=} \mathrm{brd0}\\
\mathrm{brd}^{\ell+1} &\stackrel{\mathrm{def}}{=} \mathrm{brdS}\left(\mathrm{brd}^\ell, \mathrm{midL}\left(\mathrm{dstL}\left(\mathrm{ins}^\ell\right), \mathrm{staL}\left(\mathrm{brd}^\ell, \mathrm{dstL}\left(\mathrm{ins}^\ell\right)\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{ins}^0 &\stackrel{\mathrm{def}}{=} \mathrm{ins0}\\
\mathrm{ins}^{\ell+1} &\stackrel{\mathrm{def}}{=} \mathrm{insS}\left(\mathrm{ins}^\ell, \mathrm{dstL}\left(\mathrm{ins}^\ell\right), \mathrm{staL}\left(\mathrm{brd}^\ell, \mathrm{dstL}\left(\mathrm{ins}^\ell\right)\right)\right)
\end{aligned}$$

$$\mathrm{dst}^\ell \stackrel{\mathrm{def}}{=} \mathrm{dstL}(\mathrm{ins}^\ell)$$

$$\mathrm{sta}^\ell \stackrel{\mathrm{def}}{=} \mathrm{staL}(\mathrm{brd}^\ell, \mathrm{dst}^\ell)$$

$$\mathrm{mid}^\ell \stackrel{\mathrm{def}}{=} \mathrm{midL}(\mathrm{dst}^\ell, \mathrm{sta}^\ell)$$

where $f(g_1, \ldots, g_k)$ denotes the field $(t, c) \mapsto f(t, c, g_1, \ldots, g_k)$.

In particular, we obtain the equivalence between the respective boolean and proposition fields, and the specification of dst:

Lemma 1.10 (Equivalence Lemma)**.**

$$\begin{aligned}
\forall \ell t c, \mathrm{Brd}_t^\ell(c) &\Leftrightarrow \mathrm{brd}_t^\ell(c) = \mathrm{true} \\
\forall \ell t c, \mathrm{Ins}_t^\ell(c) &\Leftrightarrow \mathrm{ins}_t^\ell(c) = \mathrm{true} \\
\forall \ell t c, \mathrm{Sta}_t^\ell(c) &\Leftrightarrow \mathrm{sta}_t^\ell(c) = \mathrm{true} \\
\forall \ell t c, \mathrm{Mid}_t^\ell(c) &\Leftrightarrow \mathrm{mid}_t^\ell(c) = \mathrm{true}
\end{aligned}$$

Lemma 1.11 (Distance Field)**.**

$$\begin{aligned}
\mathrm{dst}_0^\ell(c) &= 0 \\
\mathrm{Ins}_{t+1}^\ell(c) \Rightarrow \mathrm{dst}_{t+1}^\ell(c) &= 1 + \min\left(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\right) \\
\neg\,\mathrm{Ins}_{t+1}^\ell(c) \Rightarrow \mathrm{dst}_{t+1}^\ell(c) &= 0
\end{aligned} \tag{6}$$

2 TECHNICAL LEMMAS

Lemma 2.1 (Local Distance)**.**

$$\forall \ell t c, \mathrm{dst}_{t+1}^\ell(c) \leq 1 + \min\left(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\right)$$

Proof. Let ℓ, t and c. By case :

- If $\mathrm{Ins}_{t+1}^\ell(c)$ then (6) the equality holds, so does the inequality.
- If $\neg\,\mathrm{Ins}_{t+1}^\ell(c)$ then (6) $\mathrm{dst}_{t+1}^\ell(c) = 0$, so the inequality holds.

□

Lemma 2.2 (Middle Distance)**.**

$$\forall \ell t c, \mathrm{Mid}_{t+1}^\ell(c) \Rightarrow \mathrm{dst}_{t+1}^\ell(c) \geq \max\left(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\right)$$

Proof. Let ℓ, t and c. By using (5), $\mathrm{Mid}_{t+1}^\ell(c)$ implies two cases:

$$\mathrm{dst}_{t+1}^\ell(c) > \max\left(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\right)$$

$$\mathrm{dst}_{t+1}^\ell(c) = \max\left(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\right)$$

and the result holds in every cases. □

Remark. We could use the previous lemma to simplify the proof of the following.

Lemma 2.3 (Brd and Ins are exclusive)**.**

$$\forall \ell t c, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{False}$$

Proof. The proof is made by induction on ℓ:

- If $\ell = 0$ then $\mathrm{Brd}_t^\ell(c)$ implies (2) that $1 = c$ or $c = n$, and $\mathrm{Ins}_t^\ell(c)$ implies (3) that $1 < c < n$, hence the contradiction.

- We assume that:

$$\forall t c, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{False} \qquad (IH_\ell)$$

  Let t and c, and we assume that:

$$\mathrm{Brd}_t^{\ell+1}(c) \qquad (H_{\mathrm{Brd}})$$

$$\mathrm{Ins}_t^{\ell+1}(c) \qquad (H_{\mathrm{Ins}})$$

  The proof of False is made by case on t :

  – If $t = 0$ then (3) $\mathrm{Ins}_t^{\ell+1}(c)$ is False, and is assumed.

  – If $t = t' + 1$, H_{Ins} implies (3) that:

$$\mathrm{Ins}_{t'+1}^\ell(c) \qquad (H_{\mathrm{Ins}}2)$$

$$\mathrm{dst}_{t'+1}^\ell(c) < \mathrm{dst}_{t'}^\ell(c-1) \vee \mathrm{dst}_{t'+1}^\ell(c) < \mathrm{dst}_{t'}^\ell(c-1) \quad (H_{\mathrm{dst}})$$

  H_{Brd} implies (2) that $\mathrm{Brd}_{t'+1}^\ell(c) \vee \mathrm{Mid}_{t'+1}^\ell(c)$, so the proof is made by case:

  * If $\mathrm{Brd}_{t'+1}^\ell(c)$, because $H_{\mathrm{Ins}}2$, we have False by using IH_ℓ.
  * If $\mathrm{Mid}_{t'+1}^\ell(c)$, then by lemma 2.2:

$$\mathrm{dst}_{t'+1}^\ell(c) \geq \max\Big(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\Big)$$

  therefore $\mathrm{dst}_{t'+1}^\ell(c) \geq \mathrm{dst}_{t'}^\ell(c-1)$ and $\mathrm{dst}_{t'+1}^\ell(c) \geq \mathrm{dst}_{t'}^\ell(c+1)$, which contradicts H_{dst}.

□

Lemma 2.4 (Distance of a Border)**.**

$$\forall \ell t c, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) = 0$$

Proof. Assuming that $\mathrm{Brd}_t^\ell(c)$, by using the lemma 2.3, we have that $\neg \mathrm{Ins}_t^\ell(c)$. Therefore (6) $\mathrm{dst}_t^\ell(c) = 0$. $\square$

Lemma 2.5 (Middles have stable neighbours)**.**

$$\forall \ell t c, \mathrm{Mid}_{t+1}^\ell(c) \Rightarrow \mathrm{Sta}_t^\ell(c-1) \wedge \mathrm{Sta}_t^\ell(c+1)$$

Proof. The result is obtained by hypothesis on the two cases (5) of $\mathrm{Mid}_{t+1}^\ell(c)$. $\square$

Remark. We could use the previous lemma (introduced lately during the redaction) to simplify some proofs.

Lemma 2.6 (A middle is stable)**.**

$$\forall \ell t c, \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{Sta}_t^\ell(c)$$

Proof. Let ℓ. The proof is made by case on t:

- If $t = 0$, let c. By (5), $\mathrm{Mid}_0^\ell(c)$ is False, so the implication holds.
- Else, we prove $\mathrm{Sta}_t^\ell(c)$ by case (5) on the hypothesis $\mathrm{Mid}_t^\ell(c)$:
  - In the first case we assume:

$$\mathrm{dst}_{t+1}^\ell(c) > \max\Big(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\Big) \tag{\textit{Hd}}$$

$$\mathrm{Sta}_t^\ell(c-1) \tag{\textit{HSL}}$$

$$\mathrm{Sta}_t^\ell(c+1) \tag{\textit{HSR}}$$

Hd implies that:

$$\begin{aligned} \mathrm{dst}_{t+1}^\ell(c) &\geq 1 + \max\Big(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\Big) \\ &\geq 1 + \mathrm{dst}_t^\ell(c-1) \end{aligned}$$

And the lemma 2.1 implies that:

$$\begin{aligned} \mathrm{dst}_{t+1}^\ell(c) &\leq 1 + \min\Big(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\Big) \\ &\leq 1 + \mathrm{dst}_t^\ell(c-1) \end{aligned}$$

So $\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c-1)$. But *HSL*, therefore (4) $\mathrm{Sta}_t^\ell(c)$.

- In the second case, $\mathrm{Sta}_t^\ell(c)$ is obtained by hypothesis.

□

Lemma 2.7 (A stable cell with dst = 0 is a border)**.**

$$\forall \ell t c, \mathrm{Sta}_t^\ell(c) \wedge \mathrm{dst}_t^\ell(c) = 0 \Rightarrow \mathrm{Brd}_t^\ell(c)$$

Proof. Let ℓ. The proof is made by case on t:

- If $t = 0$, let c. We assume that $\mathrm{Sta}_0^\ell(c)$ and $\mathrm{dst}_0^\ell(c) = 0$. $\mathrm{Brd}_0^\ell(c)$ is obtained (4) with the hypothesis $\mathrm{Sta}_0^\ell(c)$.

- Else, let c. We assume that $\mathrm{Sta}_{t+1}^\ell(c)$ and $\mathrm{dst}_{t+1}^\ell(c) = 0$. The proof is made by case (4) on the hypothesis $\mathrm{Sta}_{t+1}^\ell(c)$:
  - In the first case $\mathrm{Brd}_{t+1}^\ell(c)$ is obtained by hypothesis.
  - In the second case we have $\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c-1)$, which contradicts $\mathrm{dst}_{t+1}^\ell(c) = 0$.
  - In the second case we have $\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c+1)$, which contradicts $\mathrm{dst}_{t+1}^\ell(c) = 0$.

□

Corollary 2.8 (A non-border Middle has a distance > 0)**.**

$$\forall \ell t c, \neg \mathrm{Brd}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) > 0$$

Proof. By using the contraposition of the lemma 2.7 on the hypothesis $\neg \mathrm{Brd}_t^\ell(c)$ we have $\neg \mathrm{Sta}_t^\ell(c)$ or $\mathrm{dst}_t^\ell(c) \neq 0$.

But by using the lemma 2.6 on the hypothesis $\mathrm{Mid}_t^\ell(c)$ we have $\mathrm{Sta}_t^\ell(c)$.

So $\mathrm{dst}_t^\ell(c) > 0$. □

Lemma 2.9 (At layer 0, the cells end up being awaken)**.**

$$\exists t, \forall c, \mathrm{Inp}_t(c)$$

Proof. By axiom 1.2, there exists at least one general. Therefore, the input field propagates until every cell is awaken. □

Remark. An explicit formula could be found, using the initial position of the generals.

3 MONOTONICITY

In this section we prove monotonicity properties for the fields, which means that if the property is verified for a given t, then this property is verified for every $t' \geq t$.

Lemma 3.1 (Inp is monotone)**.**

$$\forall \ell t c, \mathrm{Inp}_t^\ell(c) \Rightarrow \mathrm{Inp}_{t+1}^\ell(c)$$

Proof. Let ℓ, t and c.

The hypothesis $\mathrm{Inp}_t^\ell(c)$ implies $\mathrm{Inp}_{t+1}^\ell(c)$ by using the equation (1). $\square$

Lemma 3.2 (Ins monotone implies dst is increasing)**.**

$$\forall \ell, \Big(\forall t c, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\Big) \Rightarrow \Big(\forall t c, \mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_{t+1}^\ell(c)\Big)$$

Proof. Let ℓ, and we assume:

$$\forall t c, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \qquad (H_{ins})$$

The proof is made by induction on t:

- If $t = 0$, then (6) $\mathrm{dst}_t^\ell(c) = 0$, therefore $\mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_{t+1}^\ell(c)$.
- We assume that:

$$\forall c, \mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_{t+1}^\ell(c) \qquad (IH_t)$$

  Let c. We proove by case that $\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$:

  – If $\mathrm{Ins}_{t+2}^\ell(c)$ then (6) $\mathrm{dst}_{t+2}^\ell(c) = 1+\min\Big(\mathrm{dst}_{t+1}^\ell(c-1), \mathrm{dst}_{t+1}^\ell(c+1)\Big)$. But by using IH_t we have that $\mathrm{dst}_t^\ell(c-1) \leq \mathrm{dst}_{t+1}^\ell(c-1)$ and $\mathrm{dst}_t^\ell(c+1) \leq \mathrm{dst}_{t+1}^\ell(c+1)$, so:

$$1 + \min\Big(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\Big) \leq \mathrm{dst}_{t+2}^\ell(c)$$

  Therefore, by using the lemma 2.1, we have $\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$.

  – If $\neg\,\mathrm{Ins}_{t+2}^\ell(c)$ then (6) $\mathrm{dst}_{t+2}^\ell(c) = 0$. Moreover, by using the contraposition of H_{ins} we have $\neg\,\mathrm{Ins}_{t+1}^\ell(c)$, so $\mathrm{dst}_{t+1}^\ell(c) = 0$ too. Therefore, in any cases, $\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$.

$\square$

Lemma 3.3 (Brd and Ins monotone implies a stable dst is constant)**.**

$$\forall \ell, \Big(\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c)\Big) \Rightarrow \Big(\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\Big)$$
$$\Rightarrow \Big(\forall tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) = \mathrm{dst}_{t+1}^\ell(c)\Big)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c) \tag{H_{Brd}}$$

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \tag{H_{Ins}}$$

We prove $\forall tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) = \mathrm{dst}_{t+1}^\ell(c)$ by induction on t:

- If $t = 0$ then (4) the hypothesis $\mathrm{Sta}_0^\ell(c)$ implies $\mathrm{Brd}_0^\ell(c)$, so according to H_{Brd} we have $\mathrm{Brd}_1^\ell(c)$ too. Therefore, according to the lemma 2.4, we have $\mathrm{dst}_0^\ell(c) = 0 = \mathrm{dst}_1^\ell(c)$.

- We assume the induction hypothesis:

  $$\forall c, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) = \mathrm{dst}_{t+1}^\ell(c) \tag{IH_t}$$

  Let c. We assume the hypothesis:

  $$\mathrm{Sta}_{t+1}^\ell(c) \tag{H_{Sta}}$$

  We prove $\mathrm{dst}_{t+1}^\ell(c) = \mathrm{dst}_{t+2}^\ell(c)$ by case (4) on H_{Sta}:

  – If $\mathrm{Brd}_{t+1}^\ell(c)$ then according to H_{Brd} we have $\mathrm{Brd}_{t+2}^\ell(c)$ too. Therefore, according to the lemma 2.4, we have $\mathrm{dst}_{t+1}^\ell(c) = 0 = \mathrm{dst}_{t+2}^\ell(c)$.

  – In that case, we have:

    $$\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c-1) \tag{H_{dst}}$$

    $$\mathrm{Sta}_t^\ell(c-1) \tag{$H_{\mathrm{Sta}}2$}$$

    Firstly, by using $H_{\mathrm{Sta}}2$ and the induction hypothesis IH_t we have $\mathrm{dst}_t^\ell(c-1) = \mathrm{dst}_{t+1}^\ell(c-1)$, so by using H_{dst}, we have :

    $$\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c-1) = 1 + \mathrm{dst}_{t+1}^\ell(c-1)$$

    Moreover, by using the lemma 2.1, we have:

    $$\begin{aligned} \mathrm{dst}_{t+2}^\ell(c) &\le 1 + \min\Big(\mathrm{dst}_{t+1}^\ell(c-1), \mathrm{dst}_{t+1}^\ell(c+1)\Big) \\ &\le 1 + \mathrm{dst}_{t+1}^\ell(c-1) \\ &\le \mathrm{dst}_{t+1}^\ell(c) \end{aligned}$$

Secondly, by using H_{Ins} and the lemma 3.2:

$$\mathrm{dst}_{t+1}^{\ell}(c) \leq \mathrm{dst}_{t+2}^{\ell}(c)$$

Therefore, we proved the equality.

– If $\mathrm{dst}_{t+1}^{\ell}(c) = 1+\mathrm{dst}_{t}^{\ell}(c+1)$ and $\mathrm{Sta}_{t}^{\ell}(c+1)$, the proof is similar to the previous case.

□

Lemma 3.4 (Brd and Ins monotone implies Sta monotone)**.**

$$\forall \ell, \left(\forall tc, \mathrm{Brd}_{t}^{\ell}(c) \Rightarrow \mathrm{Brd}_{t+1}^{\ell}(c)\right) \Rightarrow \left(\forall tc, \mathrm{Ins}_{t}^{\ell}(c) \Rightarrow \mathrm{Ins}_{t+1}^{\ell}(c)\right)$$

$$\Rightarrow \left(\forall tc, \mathrm{Sta}_{t}^{\ell}(c) \Rightarrow \mathrm{Sta}_{t+1}^{\ell}(c)\right)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}_{t}^{\ell}(c) \Rightarrow \mathrm{Brd}_{t+1}^{\ell}(c) \qquad (H_{\mathrm{Brd}})$$

$$\forall tc, \mathrm{Ins}_{t}^{\ell}(c) \Rightarrow \mathrm{Ins}_{t+1}^{\ell}(c) \qquad (H_{\mathrm{Ins}})$$

We prove $\forall tc, \mathrm{Sta}_{t}^{\ell}(c) \Rightarrow \mathrm{Sta}_{t+1}^{\ell}(c)$ by induction on t:

- If $t = 0$ then (4) the hypothesis $\mathrm{Sta}_{0}^{\ell}(c)$ implies $\mathrm{Brd}_{0}^{\ell}(c)$, so according to H_{Brd} we have $\mathrm{Brd}_{1}^{\ell}(c)$ too. Therefore, we have (4) the first case of $\mathrm{Sta}_{1}^{\ell}(c)$.

- We assume the induction hypothesis:

$$\forall c, \mathrm{Sta}_{t}^{\ell}(c) \Rightarrow \mathrm{Sta}_{t+1}^{\ell}(c) \qquad (IH_t)$$

Let c. We assume the hypothesis:

$$\mathrm{Sta}_{t+1}^{\ell}(c) \qquad (H_{\mathrm{Sta}})$$

We prove $\mathrm{Sta}_{t+2}^{\ell}(c)$ by case (4) on H_{Sta}:

– If $\mathrm{Brd}_{t+1}^{\ell}(c)$ then according to H_{Brd} we have $\mathrm{Brd}_{t+2}^{\ell}(c)$ too. Therefore, we have (4) the first case of $\mathrm{Sta}_{t+2}^{\ell}(c)$.

– In that case, we have:

$$\mathrm{dst}_{t+1}^{\ell}(c) = 1 + \mathrm{dst}_{t}^{\ell}(c-1) \qquad (H_{\mathrm{dst}})$$

$$\mathrm{Sta}_t^\ell(c-1) \tag{$H_{\mathrm{Sta}}2$}$$

By using H_{Brd}, H_{Ins} and the lemma 3.3, $H_{\mathrm{Sta}}2$ implies that:

$$\mathrm{dst}_t^\ell(c-1) = \mathrm{dst}_{t+1}^\ell(c-1) \tag{H}$$

Firstly, by using the lemma 2.1 then H then H_{dst}, we have:

$$\begin{aligned}
\mathrm{dst}_{t+2}^\ell(c) &\leq 1 + \min\left(\mathrm{dst}_{t+1}^\ell(c-1), \mathrm{dst}_{t+1}^\ell(c+1)\right)\\
&\leq 1 + \mathrm{dst}_{t+1}^\ell(c-1)\\
&= 1 + \mathrm{dst}_t^\ell(c-1)\\
&= \mathrm{dst}_{t+1}^\ell(c)
\end{aligned}$$

Secondly, by using H_{Ins} and the lemma 3.2, we have:

$$\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$$

Therefore $\mathrm{dst}_{t+1}^\ell(c) = \mathrm{dst}_{t+2}^\ell(c)$. So, by using H_{dst} then H:

$$\begin{aligned}
\mathrm{dst}_{t+2}^\ell(c) &= \mathrm{dst}_{t+1}^\ell(c)\\
&= 1 + \mathrm{dst}_t^\ell(c-1)\\
&= 1 + \mathrm{dst}_{t+1}^\ell(c-1)
\end{aligned}$$

Moreover, by using $H_{\mathrm{Sta}}2$ and the induction hypothesis IH_t we have $\mathrm{Sta}_{t+1}^\ell(c-1)$. Therefore (4) we proved $\mathrm{Sta}_{t+2}^\ell(c)$.

– If $\mathrm{dst}_{t+1}^\ell(c) = 1+\mathrm{dst}_t^\ell(c+1)$ and $\mathrm{Sta}_t^\ell(c+1)$, the proof is similar to the previous case.

□

Lemma 3.5 (Brd and Ins monotone implies Mid monotone)**.**

$$\forall \ell, \left(\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c)\right) \Rightarrow \left(\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\right)$$

$$\Rightarrow \left(\forall tc, \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{Mid}_{t+1}^\ell(c)\right)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c) \tag{H_{Brd}}$$

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \tag{H_{Ins}}$$

We prove $\forall tc, \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{Mid}_{t+1}^\ell(c)$ by case on t:

- If $t = 0$ then (5) $\mathrm{Mid}_t^\ell(c)$ is False, so the implication holds.
- If $t = t' + 1$, let c, and we assume the hypothesis:

$$\mathrm{Mid}_{t'+1}^\ell(c) \tag{H_{Mid}}$$

  We prove $\mathrm{Mid}_{t'+2}^\ell(c)$ by case (5) on H_{Mid}:

  - In the first case, we have:

$$\mathrm{dst}_{t'+1}^\ell(c) > \max\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right) \tag{H_{dst}}$$

$$\mathrm{Sta}_{t'}^\ell(c-1) \tag{$H_{\mathrm{Sta}}L$}$$

$$\mathrm{Sta}_{t'}^\ell(c+1) \tag{$H_{\mathrm{Sta}}R$}$$

    By using H_{Brd}, H_{Ins} and the lemma 3.3:

    * $H_{\mathrm{Sta}}L$ implies that $\mathrm{dst}_{t'}^\ell(c-1) = \mathrm{dst}_{t'+1}^\ell(c-1)$
    * $H_{\mathrm{Sta}}R$ implies that $\mathrm{dst}_{t'}^\ell(c+1) = \mathrm{dst}_{t'+1}^\ell(c+1)$

    Therefore, we have:

$$\max\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right) = \max\left(\mathrm{dst}_{t'+1}^\ell(c-1), \mathrm{dst}_{t'+1}^\ell(c+1)\right) \tag{$H_{\max}$}$$

    So, by using H_{Ins} and the lemma 3.2, then H_{dst}, then $H_{\max}$, we have:

$$\begin{aligned}\mathrm{dst}_{t'+2}^\ell(c) &\geq \mathrm{dst}_{t'+1}^\ell(c)\\ &> \max\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right)\\ &= \max\left(\mathrm{dst}_{t'+1}^\ell(c-1), \mathrm{dst}_{t'+1}^\ell(c+1)\right)\end{aligned}$$

    Moreover, by using H_{Brd}, H_{Ins} and the lemma 3.4:

    * $H_{\mathrm{Sta}}L$ implies that $\mathrm{Sta}_{t'+1}^\ell(c-1)$
    * $H_{\mathrm{Sta}}R$ implies that $\mathrm{Sta}_{t'+1}^\ell(c+1)$

    Therefore, we have the left part of $\mathrm{Mid}_{t'+2}^\ell(c)$.

  - In the second case, we have:

$$\mathrm{dst}_{t'+1}^\ell(c) = \max\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right) \tag{H_{dst}}$$

$$\mathrm{Sta}_{t'}^\ell(c-1) \tag{$H_{\mathrm{Sta}}L$}$$

$$\mathrm{Sta}_{t'}^\ell(c) \tag{$H_{\mathrm{Sta}}C$}$$

$$\mathrm{Sta}_{t'}^\ell(c+1) \tag{$H_{\mathrm{Sta}}R$}$$

    By using H_{Brd}, H_{Ins} and the lemma 3.4:

* $H_{\mathrm{Sta}}L$ implies that $\mathrm{Sta}^{\ell}_{t'+1}(c-1)$
* $H_{\mathrm{Sta}}C$ implies that $\mathrm{Sta}^{\ell}_{t'+1}(c)$
* $H_{\mathrm{Sta}}R$ implies that $\mathrm{Sta}^{\ell}_{t'+1}(c+1)$

Therefore, to obtain the right part of $\mathrm{Mid}^{\ell}_{t'+2}(c)$, it remains only to prove that $\mathrm{dst}^{\ell}_{t'+2}(c) = \max\Big(\mathrm{dst}^{\ell}_{t'+1}(c-1), \mathrm{dst}^{\ell}_{t'+1}(c+1)\Big)$.

By using H_{Brd}, H_{Ins} and the lemma 3.3, $\mathrm{Sta}^{\ell}_{t'+1}(c)$ implies that:

$$\mathrm{dst}^{\ell}_{t'+1}(c) = \mathrm{dst}^{\ell}_{t'+2}(c) \tag{$H_{\mathrm{dst}}2$}$$

By using H_{Brd}, H_{Ins} and the lemma 3.3:

* $H_{\mathrm{Sta}}L$ implies that $\mathrm{dst}^{\ell}_{t'}(c-1) = \mathrm{dst}^{\ell}_{t'+1}(c-1)$
* $H_{\mathrm{Sta}}R$ implies that $\mathrm{dst}^{\ell}_{t'}(c+1) = \mathrm{dst}^{\ell}_{t'+1}(c+1)$

Therefore, we have:

$$\max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big) = \max\Big(\mathrm{dst}^{\ell}_{t'+1}(c-1), \mathrm{dst}^{\ell}_{t'+1}(c+1)\Big) \tag{$H_{\max}$}$$

So, by using $H_{\mathrm{dst}}2$, then H_{dst}, then $H_{\max}$, we have:

$$\begin{aligned}
\mathrm{dst}^{\ell}_{t'+2}(c) &= \mathrm{dst}^{\ell}_{t'+1}(c) \\
&= \max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big) \\
&= \max\Big(\mathrm{dst}^{\ell}_{t'+1}(c-1), \mathrm{dst}^{\ell}_{t'+1}(c+1)\Big)
\end{aligned}$$

Therefore, we have the right part of $\mathrm{Mid}^{\ell}_{t'+2}(c)$.

□

Lemma 3.6 (Brd^{ℓ} and Ins^{ℓ} monotone implies $\mathrm{Brd}^{\ell+1}$ monotone)**.**

$$\forall \ell, \Big(\forall tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell}_{t+1}(c)\Big) \Rightarrow \Big(\forall tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t+1}(c)\Big)$$

$$\Rightarrow \Big(\forall tc, \mathrm{Brd}^{\ell+1}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell+1}_{t+1}(c)\Big)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell}_{t+1}(c) \tag{H_{Brd}}$$

$$\forall tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t+1}(c) \tag{H_{Ins}}$$

Let t and c. We prove $\mathrm{Brd}^{\ell+1}_{t+1}(c)$ by case (2) on the hypothesis $\mathrm{Brd}^{\ell+1}_{t}(c)$:

- In the first case, we have $\mathrm{Brd}_t^\ell(c)$, so by using H_{Brd} we have $\mathrm{Brd}_{t+1}^\ell(c)$. Therefore (2), we proved the left part of $\mathrm{Brd}_{t+1}^{\ell+1}(c)$.
- In the second case, we have $\mathrm{Mid}_t^\ell(c)$.
  So, by using H_{Brd}, H_{Ins} and the lemma 3.5 we have $\mathrm{Mid}_{t+1}^\ell(c)$.
  Therefore (2), we proved the right part of $\mathrm{Brd}_{t+1}^{\ell+1}(c)$.

□

Lemma 3.7 (Brd^ℓ and Ins^ℓ monotone implies $\mathrm{Ins}^{\ell+1}$ monotone)**.**

$$\forall \ell, \left(\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c)\right) \Rightarrow \left(\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\right)$$

$$\Rightarrow \left(\forall tc, \mathrm{Ins}_t^{\ell+1}(c) \Rightarrow \mathrm{Ins}_{t+1}^{\ell+1}(c)\right)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c) \qquad (H_{\mathrm{Brd}})$$

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \qquad (H_{\mathrm{Ins}})$$

We prove $\mathrm{Ins}_t^{\ell+1}(c) \Rightarrow \mathrm{Ins}_{t+1}^{\ell+1}(c)$ by case on t:

- If $t = 0$ then (3) $\mathrm{Ins}_t^{\ell+1}(c)$ is False, so the implication holds.
- If $t = t' + 1$, let c. The hypothesis $\mathrm{Ins}_{t'+1}^{\ell+1}(c)$ implies (3):

  $$\mathrm{Ins}_{t'+1}^\ell(c) \qquad (H_{\mathrm{Ins}}2)$$

  $$\mathrm{Sta}_{t'+1}^\ell(c) \qquad (H_{\mathrm{Sta}})$$

  $$\mathrm{dst}_{t'+1}^\ell(c) < \mathrm{dst}_{t'}^\ell(c-1) \vee \mathrm{dst}_{t'+1}^\ell(c) < \mathrm{dst}_{t'}^\ell(c-1) \qquad (H_{\mathrm{dst}})$$

  By using H_{Ins}, $H_{\mathrm{Ins}}2$ implies that $\mathrm{Ins}_{t'+2}^\ell(c)$.

  Moreover, by using H_{Brd}, H_{Ins} and the lemma 3.4, H_{Sta} implies that $\mathrm{Sta}_{t'+2}^\ell(c)$.

  Therefore, to obtain $\mathrm{Ins}_{t'+2}^{\ell+1}(c)$, it remains only to prove that $\mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1) \vee \mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1)$.

  Notice that by using H_{Brd}, H_{Ins} and the lemma 3.3, H_{Sta} implies that:

  $$\mathrm{dst}_{t'+1}^\ell(c) = \mathrm{dst}_{t'+2}^\ell(c) \qquad (H)$$

  We prove $\mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1) \vee \mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1)$ by case on H_{dst}:

– In the first case, we have $\mathrm{dst}^{\ell}_{t'+1}(c) < \mathrm{dst}^{\ell}_{t'}(c-1)$.

So, by using H, then the case hypothesis, then H_{Ins} and the lemma 3.2, we have:

$$\begin{aligned}\mathrm{dst}^{\ell}_{t'+2}(c) &= \mathrm{dst}^{\ell}_{t'+1}(c)\\ &< \mathrm{dst}^{\ell}_{t'}(c-1)\\ &\leq \mathrm{dst}^{\ell}_{t'+1}(c-1)\end{aligned}$$

Therefore, we proved the left part of $\mathrm{dst}^{\ell}_{t'+2}(c) < \mathrm{dst}^{\ell}_{t'+1}(c-1) \vee \mathrm{dst}^{\ell}_{t'+2}(c) < \mathrm{dst}^{\ell}_{t'+1}(c-1)$.

– The case $\mathrm{dst}^{\ell}_{t'+1}(c) < \mathrm{dst}^{\ell}_{t'}(c+1)$ is similar, and proves the right part of $\mathrm{dst}^{\ell}_{t'+2}(c) < \mathrm{dst}^{\ell}_{t'+1}(c-1) \vee \mathrm{dst}^{\ell}_{t'+2}(c) < \mathrm{dst}^{\ell}_{t'+1}(c-1)$.

□

Proposition 3.8 (Brd and Ins are monotone)**.**

$$\forall \ell, \Big(\forall tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell}_{t+1}(c)\Big) \wedge \Big(\forall tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t+1}(c)\Big)$$

Proof. The proof is made by induction on ℓ:

- If $\ell = 0$, we prove the two parts separately:

  – Let t and c. The hypothesis $\mathrm{Brd}^{0}_{t}(c)$ implies (2) that $\mathrm{Inp}_{t}(c)$ and $1 = c \vee c = n$.

  So, by using the lemma 3.1, we have $\mathrm{Inp}_{t+1}(c)$ and $1 = c \vee c = n$. Therefore (2) we proved that $\mathrm{Brd}^{0}_{t+1}(c)$.

  – Let t and c. The hypothesis $\mathrm{Ins}^{0}_{t}(c)$ implies (3) that $\mathrm{Inp}_{t}(c)$ and $1 < c < n$.

  So, by using the lemma 3.1, we have $\mathrm{Inp}_{t+1}(c)$ and $1 < c < n$. Therefore (3) we proved that $\mathrm{Ins}^{0}_{t+1}(c)$.

- We assume the induction hypothesis:

$$\forall tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell}_{t+1}(c) \qquad (IH^{\ell}_{\mathrm{Brd}})$$

$$\forall tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t+1}(c) \qquad (IH^{\ell}_{\mathrm{Ins}})$$

  By using IH^{ℓ}_{Brd}, IH^{ℓ}_{Ins} and the lemma 3.6, we have:

$$\forall tc, \mathrm{Brd}^{\ell+1}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell+1}_{t+1}(c)$$

By using IH^{ℓ}_{Brd}, IH^{ℓ}_{Ins} and the lemma 3.7, we have:

$$\forall tc, \mathrm{Ins}^{\ell+1}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell+1}_{t+1}(c)$$

Therefore, we proved the induction step.

□

Corollary 3.9 (Brd is monotone)**.**

$$\forall \ell tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Brd}^{\ell}_{t'}(c)\Big)$$

Proof. Let ℓ, t and c. We assume the hypothesis $\mathrm{Brd}^{\ell}_{t}(c)$.
Let t'. We prove $\mathrm{Brd}^{\ell}_{t'}(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{Brd}^{\ell}_{t'}(c)$ by hypothesis.
- If $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{Brd}^{\ell}_{t''}(c)$, then by using the left part of the proposition 3.8 we have $\mathrm{Brd}^{\ell}_{t''+1}(c)$. Therefore $\mathrm{Brd}^{\ell}_{t'}(c)$.

□

Corollary 3.10 (Ins is monotone)**.**

$$\forall \ell tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Ins}^{\ell}_{t'}(c)\Big)$$

Proof. The proof is similar to the previous one, and uses the right part of the proposition 3.8. □

Corollary 3.11 (Sta is monotone)**.**

$$\forall \ell tc, \mathrm{Sta}^{\ell}_{t}(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Sta}^{\ell}_{t'}(c)\Big)$$

Proof. Let ℓ, t and c. We assume the hypothesis $\mathrm{Sta}^{\ell}_{t}(c)$.
Let t'. We prove $\mathrm{Sta}^{\ell}_{t'}(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{Sta}^{\ell}_{t'}(c)$ by hypothesis.
- If $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{Sta}^{\ell}_{t''}(c)$, then by using both parts of the proposition 3.8 and the lemma 3.4 we have $\mathrm{Sta}^{\ell}_{t''+1}(c)$.

  Therefore $\mathrm{Sta}^{\ell}_{t'}(c)$.

□

Corollary 3.12 (Mid is monotone)**.**

$$\forall \ell t c, \mathrm{Mid}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Mid}_{t'}^\ell(c)\Big)$$

Proof. The proof is similar to the previous one, and uses both parts of the proposition 3.8 and the lemma 3.5. □

Corollary 3.13 (dst is increasing)**.**

$$\forall \ell t c t', t' \geq t \Rightarrow \mathrm{dst}_{t'}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$$

Proof. Let ℓ, t, c and t'.

We prove $\mathrm{dst}_{t'}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{dst}_{t'}^\ell(c) = \mathrm{dst}_t^\ell(c)$, therefore $\mathrm{dst}_{t'}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$.
- In that case $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{dst}_{t''}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$.

  Therefore, by using the right part of the proposition 3.8 and the lemma 3.2, then the hypothesis, we have:

$$\begin{aligned} \mathrm{dst}_{t'}^\ell(c) &= \mathrm{dst}_{t''+1}^\ell(c) \\ &\geq \mathrm{dst}_{t''}^\ell(c) \\ &\geq \mathrm{dst}_t^\ell(c) \end{aligned}$$

□

Corollary 3.14 (A stable dst is constant)**.**

$$\forall \ell t c, \mathrm{Sta}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{dst}_{t'}^\ell(c) = \mathrm{dst}_t^\ell(c)\Big)$$

Proof. Let ℓ, t and c. We assume the hypothesis $\mathrm{Sta}_t^\ell(c)$.

Let t'. We prove $\mathrm{Brd}_t^\ell(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{dst}_{t'}^\ell(c) = \mathrm{dst}_t^\ell(c)$.
- In that case $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{dst}_{t''}^\ell(c) = \mathrm{dst}_t^\ell(c)$.

  By using the hypotheses $\mathrm{Sta}_t^\ell(c)$ and $t'' \geq t$, and the lemma 3.11, we have $\mathrm{Sta}_{t''}^\ell(c)$.

  So, by using both parts of the proposition 3.8 and the lemma 3.3 we have $\mathrm{dst}_{t''}^\ell(c) = \mathrm{dst}_{t''+1}^\ell(c)$. Therefore:

$$\begin{aligned} \mathrm{dst}_{t'}^\ell(c) &= \mathrm{dst}_{t''+1}^\ell(c) \\ &= \mathrm{dst}_{t''}^\ell(c) \\ &= \mathrm{dst}_t^\ell(c) \end{aligned}$$

□

The condition $b_1 + 2 \leq b_2$ ensures not only that $b_1 < b_2$, but also that there is a cell between them, because boundaries between light cones are not light cones themselves. This excludes the regions of the final layer to be called light cones, so the results of this section are only for the phase transition.

Remark. The choice to exclude the final regions can be justified by the fact that this is the first time the region alone cannot determine the middle, because a middle needs 3 cells to appear and not only 2.

In the following example, there is seven cells, and the generals are the cells 1 and 7. At $t = 5$ the cell 4 becomes the middle of the region and the evolution becomes stable. The informations leading to the middle at $t = 5$ traveled from the entire region at $t = 2$, which is the "Light Cone" of the middle. Notice that some cells may not be awaken at this time, but they are all border or inside after the first step of the Light Cone.

cells time	1	2	3	4	5	6	7
$t = 0$	0	.	.	.	.	.	0
$t = 1$	0	1	.	.	.	1	0
$\mathbf{t = 2}$	**0**	1	1	.	1	1	**0**
$t = 3$	0	**1**	1	2	1	**1**	0
$t = 4$	0	1	**2**	2	**2**	1	0
$t = 5$	0	1	2	**3**	2	1	0

So, in this example it is true that at the layer $\ell = 0$ and the date $t = 2$ the region between the borders 1 and 7 is a Light Cone for the middle to come. This will be denoted by $\mathrm{LC}_2^0(1, 7)$ in the following definition:

Definition 4.1 (Light Cones)**.**

$$\begin{aligned}\mathrm{LC}_t^\ell(b_1, b_2) \overset{\text{def}}{=} {} & b_1 + 2 \leq b_2 \wedge \mathrm{Brd}_t^\ell(b_1) \wedge \mathrm{Brd}_t^\ell(b_2) \\ & \wedge \Big(\forall c, b_1 < c < b_2 \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\Big)\end{aligned} \tag{7}$$

Corollary 4.2 (Light Cone at layer 0)**.**

$$\exists t, \mathrm{LC}_t^0(1, n)$$

Proof. Firstly, by axiom 1.1, $n > 2$.

Secondly, by using the lemma 2.9 there exists t such that for every cell c, $\mathrm{Inp}_t(c)$. So:

- We have (2) that $\mathrm{Brd}_t^0(1)$ and $\mathrm{Brd}_t^0(n)$
- We have (3) for every $1 < c < n$ that $\mathrm{Ins}_t^0(c)$. So, by using the corollary 3.10, for every $1 < c < n$ we have that $\mathrm{Ins}_{t+1}^0(c)$.

Therefore (7) $\mathrm{LC}_t^0(1, n)$. $\square$

In the following, $\frac{a}{2}$ will denote the floor function of the half : the half of a if a is even, and the half of $a - 1$ if a is odd.

Proposition 4.3 (Running of a Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}_t^\ell(b_1, b_2) \Rightarrow \forall\ 0 \leq d \leq \frac{b_2 - b_1}{2},$$

$$\mathrm{dst}_{t+d}^\ell(b_1 + d) = d \wedge \mathrm{Sta}_{t+d}^\ell(b_1 + d)$$

$$\wedge\, \mathrm{dst}_{t+d}^\ell(b_2 - d) = d \wedge \mathrm{Sta}_{t+d}^\ell(b_2 - d)$$

$$\wedge \left(\forall\ b_1 + d \leq c \leq b_2 - d, \mathrm{dst}_{t+d}^\ell(c) \geq d\right)$$

Proof. Let ℓ, b_1, b_2 and t. We assume that $\mathrm{LC}_t^\ell(b_1, b_2)$.

The proof is made by induction on d :

- In this case, $d = 0$.

  Because $\mathrm{LC}_t^\ell(b_1, b_2)$, we have that $\mathrm{Brd}_t^\ell(b_1)$ and $\mathrm{Brd}_t^\ell(b_2)$. So, by using the lemma 2.4 we have that $\mathrm{dst}_t^\ell(b_1) = 0$ and $\mathrm{dst}_t^\ell(b_2) = 0$, and by definition (4) we have that $\mathrm{Sta}_t^\ell(b_1)$ and $\mathrm{Sta}_t^\ell(b_2)$.

  Moreover, for every $b_1 \leq c \leq b_2$ we have $\mathrm{dst}_t^\ell(c) \geq 0$ because dst is an integer field.

- We assume that $d + 1 \leq \frac{b_2 - b_1}{2}$. So $d \leq \frac{b_2 - b_1}{2}$ too, and we have the induction hypothesis:

$$\mathrm{dst}_{t+d}^\ell(b_1 + d) = d \wedge \mathrm{Sta}_{t+d}^\ell(b_1 + d)$$

$$\wedge\, \mathrm{dst}_{t+d}^\ell(b_2 - d) = d \wedge \mathrm{Sta}_{t+d}^\ell(b_2 - d)$$

$$\wedge \left(\forall\ b_1 + d \leq c \leq b_2 - d, \mathrm{dst}_{t+d}^\ell(c) \geq d\right)$$

Firstly, we prove that for every $b_1 + (d+1) \leq c \leq b_2 - (d+1)$, we have :

$$\mathrm{dst}_{t+(d+1)}^\ell(c) = 1 + \min\left(\mathrm{dst}_{t+d}^\ell(c-1), \mathrm{dst}_{t+d}^\ell(c+1)\right) \qquad (H_c)$$

Indeed, if $b_1+(d+1) \leq c \leq b_2-(d+1)$ then by transitivity we have $b_1 < c < b_2$. So, because $\mathrm{LC}_t^\ell(b_1, b_2)$ we have $\mathrm{Ins}_{t+1}^\ell(c)$. So, by monotonicity (lemma 3.10) we have $\mathrm{Ins}_{t+(d+1)}^\ell(c)$. Therefore, by using the equation (6), we have that $\mathrm{dst}_{t+(d+1)}^\ell(c) = 1 + \min\left(\mathrm{dst}_{t+d}^\ell(c-1), \mathrm{dst}_{t+d}^\ell(c+1)\right)$.

The proof is made by case on c :

- In that case, $c = b_1 + (d+1)$.

  Because $d+1 \leq \frac{b_2-b_1}{2}$, we have $2d+2 \leq b_2-b_1$, so $b_1+d+2 \leq b_2-d$. So $b_1+d \leq b_1+d+2 \leq b_2-d$, and by using the induction hypothesis we have $\mathrm{dst}_{t+d}^\ell(b_1+d+2) \geq d$.

  Moreover, by using the induction hypothesis, we have $\mathrm{dst}_{t+d}^\ell(b_1+d) = d$, so $\mathrm{dst}_{t+d}^\ell(b_1+d+2) \geq \mathrm{dst}_{t+d}^\ell(b_1+d)$.

  By using H_c with $c = b_1+d+1$, we have :

  $$\begin{aligned}\mathrm{dst}_{t+(d+1)}^\ell(b_1+d+1) &= 1 + \min\left(\mathrm{dst}_{t+d}^\ell(b_1+d), \mathrm{dst}_{t+d}^\ell(b_1+d+2)\right) \\ &= 1 + \mathrm{dst}_{t+d}^\ell(b_1+d) \\ &= 1+d\end{aligned}$$

  Moreover, because $\mathrm{dst}_{t+(d+1)}^\ell(b_1+d+1) = 1 + \mathrm{dst}_{t+d}^\ell(b_1+d)$ and by induction hypothesis $\mathrm{Sta}_{t+d}^\ell(b_1+d)$, we have by definition (4) that $\mathrm{Sta}_{t+(d+1)}^\ell(b_1+d+1)$.

- The case $c = b_2-(d+1)$ is similar, by using the induction hypothesis $\mathrm{dst}_{t+d}^\ell(b_2-d) = d$ and $\mathrm{Sta}_{t+d}^\ell(b_2-d)$.

- If $b_1+(d+1) < c < b_2-(d+1)$, then we have :

  $$b_1+d < c-1 < b_2-d-2 < b_2-d$$

  $$b_1+d < b_1+d+2 < c+1 < b_2-d$$

  So, by using the induction hypothesis we have $\mathrm{dst}_{t+d}^\ell(c-1) \geq d$ and $\mathrm{dst}_{t+d}^\ell(c+1) \geq d$.

  Therefore, by using H_c we have :

  $$\begin{aligned}\mathrm{dst}_{t+(d+1)}^\ell(c) &= 1 + \min\left(\mathrm{dst}_{t+d}^\ell(c-1), \mathrm{dst}_{t+d}^\ell(c+1)\right) \\ &\geq 1 + \min(d, d) \\ &= 1+d\end{aligned}$$

□

Corollary 4.4 (End of a Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}_t^\ell(b_1, b_2) \Rightarrow \forall d \leq \frac{b_2 - b_1}{2},$$

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(b_1 + d) = d \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(b_1 + d)$$

$$\wedge\, \mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(b_2 - d) = d \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(b_2 - d)$$

Proof. By using the corollaries 3.11 and 3.14, this is a direct corollary of the previous proposition. □

Notice that for a Light Cone $\mathrm{LC}_t^\ell(b_1, b_2)$, $b_2 - b_1 + 1$ is the number of cells forming the Light Cone, boundaries included.

Corollary 4.5 (Middle of an odd Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}_t^\ell(b_1, b_2) \wedge b_2 - b_1 + 1 \,\mathrm{odd}$$

$$\Rightarrow \mathrm{Mid}^\ell_{t+\frac{b_2-b_1}{2}}(\frac{b_1 + b_2}{2})$$

Proof. Because $\mathrm{LC}_t^\ell(b_1, b_2)$ we have $b_1 + 2 \leq b_2$, so $\frac{b_2-b_1}{2} \geq 1$.

Because $b_2 - b_1 + 1$ is odd, we have :

$$\frac{b_1 + b_2 + 1}{2} = \frac{b_1 + b_2}{2}$$

$$b_1 + \left(\frac{b_2 - b_1}{2} - 1\right) = \frac{b_1 + b_2}{2} - 1$$

$$b_1 - \left(\frac{b_2 - b_1}{2} - 1\right) = \frac{b_1 + b_2}{2} + 1$$

Because $\mathrm{LC}_t^\ell(b_1, b_2)$, by using the proposition 4.3 for $d = \frac{b_2-b_1}{2} - 1$ we have:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}-1}(\frac{b_1 + b_2}{2} - 1) = \frac{b_2 - b_1}{2} - 1 \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}-1}(\frac{b_1 + b_2}{2} - 1)$$

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}-1}(\frac{b_1 + b_2}{2} + 1) = \frac{b_2 - b_1}{2} - 1 \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}-1}(\frac{b_1 + b_2}{2} + 1)$$

Because $\mathrm{LC}_t^\ell(b_1, b_2)$, by using the proposition 4.3 for $d = \frac{b_2-b_1}{2}$ we have:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(\frac{b_1 + b_2}{2}) = \frac{b_2 - b_1}{2}$$

So, by denoting $m = \frac{b_1+b_2}{2}$ we have :

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(m) > \max\left(\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}-1}(m-1), \mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}-1}(m+1)\right)$$

with $\mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}-1}(m-1)$ and $\mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}-1}(m+1)$.

Therefore by definition (5) $\mathrm{Mid}^\ell_{t+\frac{b_2-b_1}{2}}(m)$. □

Corollary 4.6 (Middles of an even Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^\ell_t(b_1, b_2) \wedge b_2 - b_1 + 1 \,\text{even}$$

$$\Rightarrow \mathrm{Mid}^\ell_{t+\frac{b_2-b_1+1}{2}}(\frac{b_1+b_2-1}{2}) \wedge \mathrm{Mid}^\ell_{t+\frac{b_2-b_1+1}{2}}(\frac{b_1+b_2+1}{2})$$

Proof. Because $\mathrm{LC}^\ell_t(b_1, b_2)$ we have $b_1 + 2 \leq b_2$, so because $b_2 - b_1$ is odd we have $\frac{b_2-b_1-1}{2} \geq 1$.

Because $b_2 - b_1 + 1$ is even, we have :

$$\frac{b_1+b_2+1}{2} = \frac{b_1+b_2-1}{2} + 1$$

$$b_1 + \left(\frac{b_2-b_1-1}{2} - 1\right) = \frac{b_1+b_2-1}{2} - 1$$

$$b_1 - \left(\frac{b_2-b_1-1}{2} - 1\right) = \frac{b_1+b_2+1}{2} + 1$$

Because $\mathrm{LC}^\ell_t(b_1, b_2)$, by using the proposition 4.3 for $d = \frac{b_2-b_1-1}{2} - 1$ we have:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1-1}{2}-1}(\frac{b_1+b_2-1}{2} - 1) = \frac{b_2-b_1-1}{2} - 1$$

$$\text{with } \mathrm{Sta}^\ell_{t+\frac{b_2-b_1-1}{2}-1}(\frac{b_1+b_2-1}{2} - 1)$$

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1-1}{2}-1}(\frac{b_1+b_2+1}{2} + 1) = \frac{b_2-b_1-1}{2} - 1$$

$$\text{with } \mathrm{Sta}^\ell_{t+\frac{b_2-b_1-1}{2}-1}(\frac{b_1+b_2+1}{2} + 1)$$

So, by monotonicity (lemmas 3.11 and 3.14), we have :

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2-1}{2} - 1) = \frac{b_2-b_1-1}{2} - 1$$

$$\text{with } \mathrm{Sta}^\ell_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2-1}{2} - 1)$$

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2+1}{2}+1) = \frac{b_2-b_1-1}{2}-1$$

$$\text{with } \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2+1}{2}+1)$$

Because $\mathrm{LC}^{\ell}_t(b_1, b_2)$, by using the proposition 4.3 for $d = \frac{b_2-b_1-1}{2}$ we have:

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2-1}{2}) = \frac{b_2-b_1-1}{2}$$

$$\text{with } \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2-1}{2})$$

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2+1}{2}) = \frac{b_2-b_1-1}{2}$$

$$\text{with } \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2+1}{2})$$

Notice that $\frac{b_2-b_1-1}{2}+1 = \frac{b_2-b_1+1}{2}$. So, by monotonicity (lemma 3.14), we have :

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(\frac{b_1+b_2-1}{2}) = \frac{b_2-b_1-1}{2}$$

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(\frac{b_1+b_2+1}{2}) = \frac{b_2-b_1-1}{2}$$

So, by denoting $m_1 = \frac{b_1+b_2-1}{2}$ and $m_2 = \frac{b_1+b_2+1}{2}$ we have :

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(m_1) = \max\left(\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1-1), \mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1+1)\right)$$

with $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1-1)$, $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1)$ and $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1+1)$.

Therefore by definition (5) $\mathrm{Mid}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(m_1)$.

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(m_2) = \max\left(\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2-1), \mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2+1)\right)$$

with $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2-1)$, $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2)$ and $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2+1)$.

Therefore by definition (5) $\mathrm{Mid}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(m_2)$. □

In every case, we have $\mathrm{Mid}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(\frac{b_1+b_2}{2}) \wedge \mathrm{Mid}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(\frac{b_1+b_2+1}{2})$, but we thought the presentation clearer by separating both cases.

Lemma 4.7 (The other cells of a Light Cone are not Middles)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^{\ell}_t(b_1, b_2) \Rightarrow \forall t' \geq t + \frac{b_2-b_1}{2}, \forall c,$$

$$\left(b_1 \leq c < \frac{b_1+b_2}{2} \vee \frac{b_1+b_2+1}{2} < c \leq b_2\right) \Rightarrow \neg\,\mathrm{Mid}^{\ell}_{t'+1}(c)$$

Proof. Let ℓ, t, b_1 and b_2. We assume the hypothesis $\mathrm{LC}_t^\ell(b_1, b_2)$.

Let $t' \geq t + \frac{b_2-b_1}{2}$ and let c be a cell.

Firstly, we prove that $c = b_1 + d$ or $c = b_2 - d$ with $0 \leq d < \frac{b_2-b_1}{2}$. The proof is made by case on c:

- If $b_1 \leq c < \frac{b_1+b_2}{2}$, $b_1 \leq c$ implies that $c = b_1 + d$. Moreover:

$$\text{We have } b_1 + d = c < \frac{b_1 + b_2}{2}$$

$$\text{Therefore } d < \frac{b_1 + b_2}{2} - b_1 = \frac{b_1 + b_2}{2} - \frac{2b_1}{2}$$

$$\text{Therefore (see remark) } d < \frac{b_1 + b_2 - 2b_1}{2} = \frac{b_2 - b_1}{2}$$

- If $\frac{b_1+b_2+1}{2} < c \leq b_2$, $c \leq b_2$ implies that $c = b_2 - d$. Moreover:

$$\text{We have } b_2 - d = c < \frac{b_1 + b_2 + 1}{2}$$

$$\text{Therefore } d < b_2 - \frac{b_1 + b_2 + 1}{2} = \frac{2b_2}{2} - \frac{b_1 + b_2 + 1}{2}$$

$$\text{Therefore (see remark) } d < \frac{2b_2 - b_1 - b_2}{2} = \frac{b_2 - b_1}{2}$$

By using the hypothesis $\mathrm{LC}_t^\ell(b_1, b_2)$ and the corollary 4.4 on $c = b_1 + d$ or $c = b_2 - d$, we have that:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(c) = d \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(c)$$

Therefore, because $t' + 1 \geq t' \geq t + \frac{b_2-b_1}{2}$, by using the lemma 3.14 we have that:

$$\mathrm{dst}^\ell_{t'+1}(c) = d \qquad (H_d)$$

Secondly, because $0 \leq d < \frac{b_2-b_1}{2}$, we have that $0 \leq d + 1 \leq \frac{b_2-b_1}{2}$.

So, by using the hypothesis $\mathrm{LC}_t^\ell(b_1, b_2)$ and the corollary 4.4 with $d + 1$, we have that:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(b_1 + (d+1)) = d + 1 \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(b_1 + (d+1)) \qquad (H_L)$$

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(b_2 - (d+1)) = d + 1 \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(b_2 - (d+1)) \qquad (H_R)$$

We prove that $d + 1 \leq \max\left(\mathrm{dst}^\ell_{t'}(c-1), \mathrm{dst}^\ell_{t'}(c+1)\right)$ by case on c:

- If $b_1 \le c < \frac{b_1+b_2}{2}$, we have $c = b_1 + d$, so $b_1 + (d+1) = c+1$.

  Therefore, by using H_L, we have that:

  $$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1}{2}}(c+1) = d+1 \wedge \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1}{2}}(c+1)$$

  So, because $t' \ge t + \frac{b_2-b_1}{2}$, by using the lemma 3.14 we have that:

  $$\mathrm{dst}^{\ell}_{t'}(c+1) = d+1$$

  Therefore: $d+1 \le \max\left(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\right)$

- If $\frac{b_1+b_2+1}{2} < c \le b_2$, we have $c = b_2 - d$, so $b_2 - (d+1) = c-1$.

  Therefore, by using H_R, we have that:

  $$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1}{2}}(c-1) = d+1 \wedge \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1}{2}}(c-1)$$

  So, because $t' \ge t + \frac{b_2-b_1}{2}$, by using the lemma 3.14 we have that:

  $$\mathrm{dst}^{\ell}_{t'}(c-1) = d+1$$

  Therefore: $d+1 \le \max\left(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\right)$

We prove $\neg\,\mathrm{Mid}^{\ell}_{t'+1}(c)$ by contradiction. If $\mathrm{Mid}^{\ell}_{t'+1}(c)$ then, by using the lemma 2.2, we have that:

$$\mathrm{dst}^{\ell}_{t'+1}(c) \ge \max\left(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\right)$$

So, by using H_d, we have that:

$$\max\left(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\right) \le d$$

But $d+1 \le \max\left(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\right)$, hence the contradiction. □

Remark. In the previous lemma there is some parity problems to fix, wich should be proven in the appendix or found in the Coq library.

The previous lemma can be generalized by using the monotonicity of the Mid field:

Corollary 4.8 (The other cells of a Light Cone are not Middles)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^{\ell}_{t}(b_1, b_2) \Rightarrow \forall t' c,$$

$$\left(b_1 \le c < \frac{b_1+b_2}{2} \vee \frac{b_1+b_2+1}{2} < c \le b_2\right) \Rightarrow \neg\,\mathrm{Mid}^{\ell}_{t'}(c)$$

Proof. The proof is made by case on t':

- If $t' \leq t + \frac{b_2 - b_1}{2}$, we prove $\neg \operatorname{Mid}^{\ell}_{t'}(c)$ by contradiction.

  We assume that $\operatorname{Mid}^{\ell}_{t'}(c)$. So, by using the lemma 3.12 we have that $\operatorname{Mid}^{\ell}_{t+\frac{b_2-b_1}{2}+1}(c)$.

  But, by using the previous lemma with $t+\frac{b_2-b_1}{2}$, we have that $\neg \operatorname{Mid}^{\ell}_{t+\frac{b_2-b_1}{2}+1}(c)$, hence the contradiction.

- If $t' \geq t + \frac{b_2-b_1}{2} + 1$

  By using the previous lemma with $t' - 1 \geq t + \frac{b_2-b_1}{2}$, we have that $\neg \operatorname{Mid}^{\ell}_{t'}(c)$.

□

Lemma 4.9 (Each true Middle comes from a Light Cone)**.**

$$\forall \ell t m, \neg \operatorname{Brd}^{\ell}_{t}(m) \wedge \operatorname{Mid}^{\ell}_{t}(m)$$

$$\Rightarrow \operatorname{LC}^{\ell}_{t-d}(m-d, m+d)$$

$$\vee \operatorname{LC}^{\ell}_{t-(d+1)}(m-(d+1), m+d)$$

$$\vee \operatorname{LC}^{\ell}_{t-(d+1)}(m-d, m+(d+1))$$

$$\textit{where } d = \operatorname{dst}^{\ell}_{t}(m)$$

Proof. To do ! I already proved it, I will recopy soon. □

Corollary 4.10 (A Middle induces a new Light Cone)**.**

$$\forall \ell t m d, \operatorname{Mid}^{\ell}_{t}(m) \wedge \operatorname{dst}^{\ell}_{t}(m) = d \wedge d \geq 2$$

$$\Rightarrow \forall t', \left(\operatorname{Brd}^{\ell}_{t'}(m-d) \Rightarrow \operatorname{LC}^{\ell+1}_{t}(m-d, m)\right)$$

$$\wedge \left(\operatorname{Brd}^{\ell}_{t'}(m+d) \Rightarrow \operatorname{LC}^{\ell+1}_{t}(m, m+d)\right)$$

Proof. Because $\operatorname{dst}^{\ell}_{t}(m) = d \geq 2$, $\operatorname{Mid}^{\ell}_{t}(m)$, by using the contraposition of the lemma 2.4, we have that $\neg \operatorname{Brd}^{\ell}_{t}(m)$.

The proof is made by case on the border. We assume $\operatorname{Brd}^{\ell}_{t'}(m-d)$, but the proof in the case $\operatorname{Brd}^{\ell}_{t'}(m+d)$ is similar.

Because $\neg \operatorname{Brd}^{\ell}_{t}(m)$ and $\operatorname{Mid}^{\ell}_{t}(m)$, by using the previous lemma we have that $\operatorname{LC}^{\ell}_{t-d}(m-d, m+d)$ or $\operatorname{LC}^{\ell}_{t-(d+1)}(m-(d+1), m+d)$ or $\operatorname{LC}^{\ell}_{t-(d+1)}(m-d, m+(d+1))$.

By definition (7), if $\mathrm{LC}^{\ell}_{t-(d+1)}(m-(d+1), m+d)$ then $\mathrm{Ins}^{\ell}_{t-d}(m-d)$. So, by case $t' \leq t-d$ or $t-d \leq t'$ and by monotonicity, we obtain a contradiction dy using the lemma 2.3. So we have :

$$\mathrm{LC}^{\ell}_{t-d}(m-d, m+d) \vee \mathrm{LC}^{\ell}_{t-(d+1)}(m-d, m+(d+1)) \qquad (H_{LC})$$

In every case, by using the lemma 4.4 we have for every $i \leq d$ that $\mathrm{dst}^{\ell}_{t}(m-d+i) = i$ and $\mathrm{Sta}^{\ell}_{t}(m-d+i)$.

So, by using the monotonicity, for every $m-d < c < m$, we have $\mathrm{dst}^{\ell}_{t+1}(c) + 1 = \mathrm{dst}^{\ell}_{t}(c+1)$.

Moreover, by using the monotonicity, $\mathrm{Sta}^{\ell}_{t+1}(c)$.

Moreover, by using H_{LC} and the definition (7) and the monotonicity, we have $\mathrm{Ins}^{\ell}_{t+1}(c)$.

Therefore, by definition (3), we have $\mathrm{Ins}^{\ell+1}_{t+1}(c)$.

Moreover, by using H_{LC} and the definition (7) and the monotonicity, we have $\mathrm{Brd}^{\ell}_{t+1}(m-d)$. So, by definition (2) $\mathrm{Brd}^{\ell+1}_{t+1}(m-d)$.

Moreover, by using H_{LC} and (the lemma 4.5 or the lemma 4.6) and the monotonicity, we have $\mathrm{Mid}^{\ell}_{t+1}(m)$. So, by definition (2) $\mathrm{Brd}^{\ell+1}_{t+1}(m)$.

Moreover, by hypothesis $d \geq 2$, so $(m-d)+2 \leq m$.

Therefore, by definition (7) we have $\mathrm{LC}^{\ell+1}_{t}(m-d, m)$. □

Lemma 4.11 (One Brd and one Mid at previous layer of a Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^{\ell+1}_{t}(b_1, b_2)$$
$$\Rightarrow \Big(\mathrm{Brd}^{\ell}_{t}(b_1) \wedge \neg\,\mathrm{Brd}^{\ell}_{t}(b_2) \wedge \mathrm{Mid}^{\ell}_{t}(b_2) \wedge \mathrm{dst}^{\ell}_{t}(b_2) = b_2 - b_1\Big)$$
$$\vee \Big(\neg\,\mathrm{Brd}^{\ell}_{t}(b_1) \wedge \mathrm{Mid}^{\ell}_{t}(b_1) \wedge \mathrm{Brd}^{\ell}_{t}(b_2) \wedge \mathrm{dst}^{\ell}_{t}(b_1) = b_2 - b_1\Big)$$

Proof. To do ! I already proved it, I will recopy soon. □

5 MIDDLES

Lemma 5.1 (Paired Middles appear at the same time with the same distance)**.**

$$\forall \ell t_1 t_2 m_1 m_2, \mathrm{Mid}^{\ell}_{t_1}(m_1) \wedge \mathrm{Mid}^{\ell}_{t_2}(m_2) \wedge (m_2 = m_1 + 1 \vee m_1 = m_2 + 1)$$
$$\Rightarrow \mathrm{Mid}^{\ell}_{t_1}(m_2) \wedge \mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)$$

Proof. We assume that $m_2 = m_1 + 1$ (the case $m_1 = m_2 + 1$ is symmetrical). For sake of simplicity, we note $c_1 = m_1 - 1$ and $c_2 = m_2 + 1$.

We assume that $t_1 \leq t_2$ and we prove the result both for t_1 and t_2. Notice that because of the middles, we have $t_1, t_2 \geq 1$.

We note $d_1 = \mathrm{dst}^\ell_{t_1}(m_1)$ and $d_2 = \mathrm{dst}^\ell_{t_2}(m_2)$, and we prove that $d_1 = d_2$:

By using the lemma 2.2 on $\mathrm{Mid}^\ell_{t_1}(m_1)$ we have that :

$$d_1 \geq \max\left(\mathrm{dst}^\ell_{t_1-1}(c_1), \mathrm{dst}^\ell_{t_1-1}(m_2)\right) \geq \mathrm{dst}^\ell_{t_1-1}(m_2)$$

By using the lemma 2.5 on $\mathrm{Mid}^\ell_{t_1}(m_1)$, we have $\mathrm{Sta}^\ell_{t_1-1}(m_2)$. So, by monotonicity (lemma 3.14), $\mathrm{dst}^\ell_{t_1-1}(m_2) = d_2$. Therefore $d_1 \geq d_2$.

By using the lemma 2.2 on $\mathrm{Mid}^\ell_{t_2}(m_2)$ we have that :

$$d_2 \geq \max\left(\mathrm{dst}^\ell_{t_2-1}(m_1), \mathrm{dst}^\ell_{t_2-1}(c_2)\right) \geq \mathrm{dst}^\ell_{t_2-1}(m_1)$$

We have two cases on $t_1 \leq t_2$:

- In the case $t_1 = t_2$, by using the lemma 2.5 on $\mathrm{Mid}^\ell_{t_2}(m_2)$, we have $\mathrm{Sta}^\ell_{t_2-1}(m_1)$. So, by monotonicity (lemma 3.14), $\mathrm{dst}^\ell_{t_2-1}(m_1) = d_1$.

- In the case $t_1 < t_2$, by using the lemma 2.6 on $\mathrm{Mid}^\ell_{t_1}(m_1)$ we have $\mathrm{Sta}^\ell_{t_1}(m_1)$. So, by monotonicity (lemma 3.14), $\mathrm{dst}^\ell_{t_2-1}(m_1) = d_1$.

In every case, we have $d_2 \geq d_1$, and because we proved $d_1 \geq d_2$, we have $d_1 = d_2$. So, in the following d_1 and d_2 will be denoted by d.

Because $\mathrm{dst}^\ell_{t_1}(m_1) = d = \mathrm{dst}^\ell_{t_1-1}(m_2)$, the middle m_1 verifies the second case of the equation (5). In particular, we have that $\mathrm{Sta}^\ell_{t_1-1}(m_1)$. So, by monotonicity (lemma 3.14) we have $\mathrm{dst}^\ell_{t_1-1}(m_1) = \mathrm{dst}^\ell_{t_1}(m_1) = d$.

We have two cases on d :

- If $\mathrm{dst}^\ell_{t_1-1}(m_2) = d = 0$, because $\mathrm{Sta}^\ell_{t_1-1}(m_2)$, by (4) we have $\mathrm{Brd}^\ell_{t_1-1}(m_2)$... (Not finished, but this case may not be necessary, because it cannot happen in the case $\ell = 0$ by axiom $n > 2$ and the definition (2) of Brd, and this lemma is only used in that case.)

- If $\mathrm{dst}^\ell_{t_1-1}(m_2) = d > 0$ (In that case, because the distance is 0 at $t = 0$, we have $t_1 - 1 > 0$, so we can write $t_1 - 2$.), because $\mathrm{Sta}^\ell_{t_1-1}(m_2)$, by (4) we have two cases:

  - $\mathrm{dst}^\ell_{t_1-1}(m_2) = 1 + \mathrm{dst}^\ell_{t_1-2}(m_1) \wedge \mathrm{Sta}^\ell_{t_1-2}(m_1)$
    In that case, because $\mathrm{Sta}^\ell_{t_1-2}(m_1)$, by monotonicity (lemma 3.14) we have $\mathrm{dst}^\ell_{t_1-2}(m_1) = \mathrm{dst}^\ell_{t_1}(m_1) = d$.
    Therefore $d = \mathrm{dst}^\ell_{t_1-1}(m_2) = 1 + \mathrm{dst}^\ell_{t_1-2}(m_1) = 1 + d$, hence the contradiction.

- $\mathrm{dst}^{\ell}_{t_1-1}(m_2) = 1 + \mathrm{dst}^{\ell}_{t_1-2}(c_2) \wedge \mathrm{Sta}^{\ell}_{t_1-2}(c_2)$.

  So $\mathrm{dst}^{\ell}_{t_1-2}(c_2) = d - 1$, and by monotonicity (lemma 3.14) we have $\mathrm{dst}^{\ell}_{t_1-1}(c_2) = d - 1$.

  Moreover, because $\mathrm{Sta}^{\ell}_{t_1-2}(c_2)$, by monotonicity (lemma 3.11) we have $\mathrm{Sta}^{\ell}_{t_1-1}(c_2)$.

  Finally, because $\mathrm{dst}^{\ell}_{t_1-1}(m_2) = d_2$ and $\mathrm{Sta}^{\ell}_{t_1-1}(m_2)$, by monotonicity (lemma 3.14) we have $\mathrm{dst}^{\ell}_{t_1}(m_2) = d = \mathrm{dst}^{\ell}_{t_1}(m_1)$.

  Therefore, we have :

  * $\mathrm{dst}^{\ell}_{t_1-1}(m_1) = d$ and $\mathrm{dst}^{\ell}_{t_1-1}(c_2) = d - 1$, so :

    $$\mathrm{dst}^{\ell}_{t_1}(m_2) = d = \max(d, d-1) = \max\left(\mathrm{dst}^{\ell}_{t_1-1}(m_1), \mathrm{dst}^{\ell}_{t_1-1}(c_2)\right)$$

  * $\mathrm{Sta}^{\ell}_{t_1-1}(m_1)$ and $\mathrm{Sta}^{\ell}_{t_1-1}(m_2)$ and $\mathrm{Sta}^{\ell}_{t_1-1}(c_2)$

  So, by the definition (5), we have $\mathrm{Mid}^{\ell}_{t_1}(m_2)$.

The result can be prove for t_2 too by using the monotonicity. □

Lemma 5.2 (A Middle has the same distance over time)**.**

$$\forall \ell t_1 t_2 m, \mathrm{Mid}^{\ell}_{t_1}(m) \wedge \mathrm{Mid}^{\ell}_{t_2}(m) \Rightarrow \mathrm{dst}^{\ell}_{t_1}(m) = \mathrm{dst}^{\ell}_{t_2}(m)$$

Proof. Two cases $t_1 \leq t_2$ and $t_2 \leq t_1$. In every case, a middle is stable, therefore the distance is the same. □

Proposition 5.3 (Middles appear at the same time with the same distance)**.**

$$\forall \ell t_1 m_1, \neg\,\mathrm{Brd}^{\ell}_{t_1}(m_1) \wedge \mathrm{Mid}^{\ell}_{t_1}(m_1)$$

$$\Rightarrow \left(\forall t_2 m_2, \mathrm{Mid}^{\ell}_{t_2}(m_2) \Rightarrow \mathrm{Mid}^{\ell}_{t_1}(m_2) \wedge \mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)\right)$$

Proof. The proof is made by induction on ℓ:

- In this case $\ell = 0$.

  Let t_1 and m_1 such that $\neg\,\mathrm{Brd}^{0}_{t_1}(m_1)$ and $\mathrm{Mid}^{0}_{t_1}(m_1)$.

  Let t_2 and m_2 such that $\mathrm{Mid}^{0}_{t_2}(m_2)$.

  Because $\ell = 0$, by using the lemma 4.2 there exists t_{LC} such that $\mathrm{LC}^{0}_{t_{\mathrm{LC}}}(1, n)$. We prove $\mathrm{Mid}^{\ell}_{t_1}(m_2)$ and $\mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)$ by case on the parity of n:

– If $n = n - 1 + 1$ is odd, then by using the corollary 4.5 we have that $\mathrm{Mid}^0_{t_{\mathrm{LC}}+\frac{n-1}{2}}(\frac{n+1}{2})$.

By contradiction, if $m_1 \neq \frac{n+1}{2}$, then by using the lemma 4.8 we have that $\neg\,\mathrm{Mid}^0_{t_1}(m_1)$, which contradicts the hypothesis $\mathrm{Mid}^0_{t_1}(m_1)$. So $m_1 = \frac{n+1}{2}$.

By contradiction, if $m_2 \neq \frac{n+1}{2}$, then by using the lemma 4.8 we have that $\neg\,\mathrm{Mid}^0_{t_2}(m_2)$, which contradicts the hypothesis $\mathrm{Mid}^0_{t_2}(m_2)$. So $m_2 = \frac{n+1}{2}$.

Therefore, $m_1 = m_2$, then by hypothesis $\mathrm{Mid}^0_{t_1}(m_2)$, and we have $\mathrm{dst}^0_{t_1}(m_1) = \mathrm{dst}^0_{t_1}(m_2)$.

– If $n = n - 1 + 1$ is even, then by using the corollary 4.6 we have that $\mathrm{Mid}^0_{t_{\mathrm{LC}}+\frac{n}{2}}(\frac{n}{2})$ and $\mathrm{Mid}^0_{t_{\mathrm{LC}}+\frac{n}{2}}(\frac{n}{2}+1)$.

By contradiction, if $m_1 \neq \frac{n}{2}$ and $m_1 \neq \frac{n}{2}+1$, then by using the lemma 4.8 we have that $\neg\,\mathrm{Mid}^0_{t_1}(m_1)$, which contradicts the hypothesis $\mathrm{Mid}^0_{t_1}(m_1)$. So $m_1 = \frac{n}{2}$ or $m_1 = \frac{n}{2}+1$.

By contradiction, if $m_2 \neq \frac{n}{2}$ and $m_2 \neq \frac{n}{2}+1$, then by using the lemma 4.8 we have that $\neg\,\mathrm{Mid}^0_{t_2}(m_2)$, which contradicts the hypothesis $\mathrm{Mid}^0_{t_2}(m_2)$. So $m_2 = \frac{n}{2}$ or $m_2 = \frac{n}{2}+1$.

The proof is made by case:

  * If $m_1 = m_2$, then by hypothesis $\mathrm{Mid}^0_{t_1}(m_2)$, and we have $\mathrm{dst}^0_{t_1}(m_1) = \mathrm{dst}^0_{t_1}(m_2)$.
  * If $m_1 \neq m_2$, then $m_2 = m_1 + 1$ or $m_1 = m_2 + 1$. So, because $\mathrm{Mid}^0_{t_1}(m_1)$ and $\mathrm{Mid}^0_{t_2}(m_2)$, by using the lemma 5.1 we have $\mathrm{Mid}^0_{t_1}(m_2)$ and $\mathrm{dst}^0_{t_1}(m_1) = \mathrm{dst}^0_{t_1}(m_2)$.

- We assume the induction hypothesis:

$$\forall t_1 m_1, \neg\,\mathrm{Brd}^{\ell}_{t_1}(m_1) \wedge \mathrm{Mid}^{\ell}_{t_1}(m_1) \qquad (IH_\ell)$$
$$\Rightarrow \Big(\forall t_2 m_2, \mathrm{Mid}^{\ell}_{t_2}(m_2) \Rightarrow \mathrm{Mid}^{\ell}_{t_1}(m_2) \wedge \mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)\Big)$$

Let t_1 and m_1 such that $\neg\,\mathrm{Brd}^{\ell+1}_{t_1}(m_1)$ and $\mathrm{Mid}^{\ell+1}_{t_1}(m_1)$, and let $d_1 = \mathrm{dst}^{\ell+1}_{t_1}(m_1)$.

By using the lemma 4.9, there exists $t'_1 = t_1 - d_1$ or $t_1 - (d_1 + 1)$, $b_1 = m_1 - d_1$ or $m_1 - (d_1 + 1)$, and $b'_1 = m_1 + d_1$ or $m_1 + (d_1 + 1)$ such that $\mathrm{LC}^{\ell+1}_{t'_1}(b_1, b'_1)$.

Notice that the case $b_1 = m_1 - (d_1 + 1)$ and $b'_1 = m_1 + (d_1 + 1)$ is excluded, so $b'_1 - b_1 = 2d_1$ or $2d_1 + 1$, but not $2d_1 + 2$.

Because $\mathrm{LC}_{t_1'}^{\ell+1}(b_1, b_1')$, by using the lemma 4.11, we have that:

$$\text{either} \quad \mathrm{Brd}_{t_1'}^{\ell}(b_1) \wedge \neg\,\mathrm{Brd}_{t_1'}^{\ell}(b_1') \wedge \mathrm{Mid}_{t_1'}^{\ell}(b_1') \wedge \mathrm{dst}_{t_1'}^{\ell}(b_1') = b_1' - b_1$$

$$\text{or} \quad \neg\,\mathrm{Brd}_{t_1'}^{\ell}(b_1) \wedge \mathrm{Mid}_{t_1'}^{\ell}(b_1) \wedge \mathrm{Brd}_{t_1'}^{\ell}(b_1') \wedge \mathrm{dst}_{t_1'}^{\ell}(b_1) = b_1' - b_1$$

We denote the border by b_1^ℓ and the middle by m_1^ℓ. In particular, we have that $\mathrm{dst}_{t_1'}^{\ell}(m_1^\ell) = b_1' - b_1 = 2d_1$ or $2d_1 + 1$.

Let t_2 and m_2 such that $\mathrm{Mid}_{t_2}^{\ell+1}(m_2)$, and let $d_2 = \mathrm{dst}_{t_2}^{\ell+1}(m_2)$.

By using the same arguments, we have that $\mathrm{LC}_{t_2'}^{\ell+1}(b_2, b_2')$, and at the previous layer we denote the border by b_2^ℓ and the middle by m_2^ℓ, with $\mathrm{dst}_{t_2'}^{\ell}(m_2^\ell) = b_2' - b_2 = 2d_2$ or $2d_2 + 1$.

Because $\neg\,\mathrm{Brd}_{t_1'}^{\ell}(m_1^\ell)$ and $\mathrm{Mid}_{t_1'}^{\ell}(m_1^\ell)$ and $\mathrm{Mid}_{t_2'}^{\ell}(m_2^\ell)$, by using the induction hypothesis IH_ℓ, we have that $\mathrm{Mid}_{t_1'}^{\ell}(m_2^\ell)$ and $\mathrm{dst}_{t_1'}^{\ell}(m_1^\ell) = \mathrm{dst}_{t_1'}^{\ell}(m_2^\ell)$.

Because $\mathrm{Mid}_{t_1'}^{\ell}(m_2^\ell)$ and $\mathrm{Mid}_{t_2'}^{\ell}(m_2^\ell)$, by using the lemma 5.2 we have that $\mathrm{dst}_{t_1'}^{\ell}(m_2^\ell) = \mathrm{dst}_{t_2'}^{\ell}(m_2^\ell)$.

Therefore $\mathrm{dst}_{t_1'}^{\ell}(m_1^\ell) = \mathrm{dst}_{t_1'}^{\ell}(m_2^\ell) = \mathrm{dst}_{t_2'}^{\ell}(m_2^\ell)$.

So, because $\mathrm{dst}_{t_1'}^{\ell}(m_1^\ell) = 2d_1$ or $2d_1+1$ and $\mathrm{dst}_{t_2'}^{\ell}(m_2^\ell) = 2d_2$ or $2d_2+1$, by using the lemma 7.1 we have that $d_1 = d_2$.

Therefore $\mathrm{dst}_{t_1}^{\ell+1}(m_1) = d_1 = d_2 = \mathrm{dst}_{t_2}^{\ell+1}(m_2)$.

It remains to prove that $\mathrm{Mid}_{t_1}^{\ell+1}(m_2)$.

Because $\neg\,\mathrm{Brd}_{t_1}^{\ell+1}(m_1)$ and $\mathrm{Mid}_{t_1}^{\ell+1}(m_1)$, by using the lemma 2.8 we have that $d_2 = d_1 = \mathrm{dst}_{t_1}^{\ell+1}(m_1) \geq 1$.

So $\mathrm{dst}_{t_1'}^{\ell}(m_2^\ell) = \mathrm{dst}_{t_2'}^{\ell}(m_2^\ell) = 2d_2$ or $2d_2 + 1 \geq 2$.

Moreover, because $\mathrm{dst}_{t_1'}^{\ell}(m_2^\ell) = \mathrm{dst}_{t_2'}^{\ell}(m_2^\ell) = b_2' - b_2$, where b_2 and b_2' are b_2^ℓ and m_2^ℓ or the reverse, we have $b_2^\ell = m_2^\ell - \mathrm{dst}_{t_1'}^{\ell}(m_2^\ell)$ or $b_2^\ell = m_2^\ell + \mathrm{dst}_{t_1'}^{\ell}(m_2^\ell)$.

Moreover, $\mathrm{Mid}_{t_1'}^{\ell}(m_2^\ell)$ and $\mathrm{Brd}_{t_2'}^{\ell}(b_2^\ell)$.

Therefore, by using the lemma 4.10, we have $\mathrm{LC}_{t_1'}^{\ell+1}(b_2, b_2')$.

We prove $\mathrm{Mid}_{t_1}^{\ell+1}(m_2)$ by case on the parity of $b_2' - b_2$:

- If $b_2' - b_2$ is even, because $b_2' - b_2 = 2d_2$ or $2d_2 + 1$, we have $b_2' - b_2 = 2d_2$ so $\frac{b_2' - b_2}{2} = d_2$.

Because $b_2' - b_2$ is even, $b_2' - b_2 + 1$ is odd. So, by using the lemma 4.5, we have that $\mathrm{Mid}^{\ell+1}_{t_1'+\frac{b_2'-b_2}{2}}(\frac{b_2+b_2'}{2})$.

In that case (by using the previous results of the lemma 4.9), we have (see remark) $t_1' = t_1 - d_1$ and $b_2 = m_2 - d_2$ and $b_2' = m_2 + d_2$, so :

$$t_1' + \frac{b_2' - b_2}{2} = t_1' + d_2 = t_1' + d_1 = t_1$$

$$\frac{b_2 + b_2'}{2} = \frac{(m_2 - d_2) + (m_2 + d_2)}{2} = \frac{2m_2}{2} = m_2$$

Therefore $\mathrm{Mid}^{\ell+1}_{t_1}(m_2)$.

– If $b_2' - b_2$ is odd, because $b_2' - b_2 = 2d_2$ or $2d_2 + 1$, we have $b_2' - b_2 = 2d_2 + 1$ so $\frac{b_2'-b_2+1}{2} = d_2 + 1$.

  Because $b_2' - b_2$ is odd, $b_2' - b_2 + 1$ is even. So, by using the lemma 4.6, we have that $\mathrm{Mid}^{\ell+1}_{t_1'+\frac{b_2'-b_2+1}{2}}(\frac{b_2+b_2'-1}{2})$ and $\mathrm{Mid}^{\ell+1}_{t_1'+\frac{b_2'-b_2+1}{2}}(\frac{b_2+b_2'+1}{2})$.

  In that case (by using the previous results of the lemma 4.9), we have (see remark) $t_1' = t_1 - (d_1 + 1)$, so:

$$t_1' + \frac{b_2' - b_2 + 1}{2} = t_1' + d_2 + 1 = t_1' + d_1 + 1 = t_1$$

  Morevover, there are two cases for b_2 and b_2':

  * $b_2 = m_2 - (d_2 + 1)$ and $b_2' = m_2 + d_2$. In that case:

$$\frac{b_2 + b_2' + 1}{2} = \frac{(m_2 - d_2 - 1) + (m_2 + d_2) + 1}{2} = \frac{2m_2}{2} = m_2$$

    Therefore, because $\mathrm{Mid}^{\ell+1}_{t_1'+\frac{b_2'-b_2+1}{2}}(\frac{b_2+b_2'-1}{2})$, we have that $\mathrm{Mid}^{\ell+1}_{t_1}(m_2)$.

  * $b_2 = m_2 - d_2$ and $b_2' = m_2 + (d_2 + 1)$. In that case:

$$\frac{b_2 + b_2' - 1}{2} = \frac{(m_2 - d_2) + (m_2 + d_2 + 1) - 1}{2} = \frac{2m_2}{2} = m_2$$

    Therefore, because $\mathrm{Mid}^{\ell+1}_{t_1'+\frac{b_2'-b_2+1}{2}}(\frac{b_2+b_2'+1}{2})$, we have that $\mathrm{Mid}^{\ell+1}_{t_1}(m_2)$.

□

Remark. The remaining problems in the previous lemma come from the fact that the cases for the form of the Light Cones "may" not be the same (in particular even or odd length) for the two middles. Maybe we should prove that this is the case anyway because at a layer ℓ the Light Cones have the same length ?

Lemma 5.4 (Middles have max distance)**.**

$$\forall \ell t m, \neg\,\mathrm{Brd}_t^\ell(m) \wedge \mathrm{Mid}_t^\ell(m) \Rightarrow \Big(\forall c, \mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_t^\ell(m)\Big)$$

Lemma 5.5 (Cells with the same distance than a Middle are Middles)**.**

$$\forall \ell t m, \neg\,\mathrm{Brd}_t^\ell(m) \wedge \mathrm{Mid}_t^\ell(m) \Rightarrow \Big(\forall c, \mathrm{dst}_t^\ell(c) = \mathrm{dst}_t^\ell(m) \Rightarrow \mathrm{Mid}_t^\ell(c)\Big)$$

Lemma 5.6 (Middles appear when each cell is stable)**.**

$$\forall \ell t m, \neg\,\mathrm{Brd}_t^\ell(m) \wedge \mathrm{Mid}_t^\ell(m) \Rightarrow \forall c, \mathrm{Sta}_t^\ell(c)$$

6 SYNCHRONIZATION

Definition 6.1 (Output Field)**.**

$$\begin{aligned} \mathrm{Out}_0^\ell(c) &\stackrel{\text{def}}{=} \text{False} \\ \mathrm{Out}_{t+1}^\ell(c) &\stackrel{\text{def}}{=} \mathrm{Brd}_t^\ell(c-1) \wedge \mathrm{Brd}_t^\ell(c) \wedge \mathrm{Brd}_t^\ell(c+1) \end{aligned} \tag{8}$$

Lemma 6.2 (The Border Field is True or False)**.**

$$\forall \ell t c, \mathrm{Brd}_t^\ell(c) \vee \neg\,\mathrm{Brd}_t^\ell(c)$$

Proof. Using the definition or the characterization of bool/Prop fields. □

Lemma 6.3 (The Output fires for every layer)**.**

$$\forall \ell t c, \mathrm{Out}_{t+1}^\ell(c) \Rightarrow \mathrm{Out}_{t+1}^{\ell+1}(c)$$

Proof. Let ℓ, t and c. The hypothesis $\mathrm{Out}_{t+1}^\ell(c)$ implies (8) that $\mathrm{Brd}_t^\ell(c-1)$ and $\mathrm{Brd}_t^\ell(c)$ and $\mathrm{Brd}_t^\ell(c+1)$

So (2) we have $\mathrm{Brd}_t^{\ell+1}(c-1)$ and $\mathrm{Brd}_t^{\ell+1}(c)$ and $\mathrm{Brd}_t^{\ell+1}(c+1)$.

Therefore (8) we proved $\mathrm{Out}_{t+1}^{\ell+1}(c)$. □

Lemma 6.4 (Three non-border Middles cannot be adjacent)**.**

$$\forall \ell t c, \neg\,\mathrm{Brd}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c-1) \wedge \mathrm{Mid}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c+1) \Rightarrow \text{False}$$

Proof. We obtain a contradiction by case on t:

- If $t = 0$ then (5) $\mathrm{Mid}_0^\ell(c)$ is False.
- Else $t = t' + 1$. By hypothesis $\neg\,\mathrm{Brd}_{t'+1}^\ell(c)$, so we can use the lemma 5.3 to prove that:

  $$\mathrm{dst}_{t'+1}^\ell(c-1) = \mathrm{dst}_{t'+1}^\ell(c) = \mathrm{dst}_{t'+1}^\ell(c+1)$$

  This distance will be denoted by d.

  By using the lemma 2.5 on $\mathrm{Mid}_{t'+1}^\ell(c)$, we have that $\mathrm{Sta}_{t'}^\ell(c-1)$ and $\mathrm{Sta}_{t'}^\ell(c+1)$. So, by using the lemma 3.14 on both we have:

  $$\begin{aligned}\mathrm{dst}_{t'}^\ell(c-1) &= \mathrm{dst}_{t'+1}^\ell(c-1) = d\\ \mathrm{dst}_{t'}^\ell(c+1) &= \mathrm{dst}_{t'+1}^\ell(c+1) = d\end{aligned}$$

  Because $\neg\,\mathrm{Brd}_{t'+1}^\ell(c)$ and $\mathrm{Mid}_{t'+1}^\ell(c)$, by using the lemma 2.8 we have $\mathrm{dst}_{t'+1}^\ell(c) > 0$. So (6):

  $$\begin{aligned}\mathrm{dst}_{t'+1}^\ell(c) &= 1 + \min\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right)\\ &= 1 + \min(d, d)\\ &= 1 + d\end{aligned}$$

  which contradicts $\mathrm{dst}_{t'+1}^\ell(c) = d$.

□

Lemma 6.5 (A non-border Middle adjacent to a Border has a distance $= 1$)**.**

$\forall \ell t c, \neg\,\mathrm{Brd}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c) \wedge \left(\mathrm{Brd}_t^\ell(c-1) \vee \mathrm{Brd}_t^\ell(c+1)\right) \Rightarrow \mathrm{dst}_t^\ell(c) = 1$

Proof. We prove the result by case on t:

- If $t = 0$ then (5) $\mathrm{Mid}_0^\ell(c)$ is False, so we get a contradiction.
- Else $t = t' + 1$. Because $\neg\,\mathrm{Brd}_{t'+1}^\ell(c)$ and $\mathrm{Mid}_{t'+1}^\ell(c)$, by using the lemma 2.8 we have $\mathrm{dst}_{t'+1}^\ell(c) > 0$. So (6):

  $$\mathrm{dst}_{t'+1}^\ell(c) = 1 + \min\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right)$$

  But $\left(\mathrm{Brd}_{t'+1}^\ell(c-1) \vee \mathrm{Brd}_{t'+1}^\ell(c+1)\right)$, so by using the lemma 2.4 we have $\left(\mathrm{dst}_{t'+1}^\ell(c-1) = 0 \vee \mathrm{dst}_{t'+1}^\ell(c+1) = 0\right)$, and by using the lemma 3.13 we have $\left(\mathrm{dst}_{t'}^\ell(c-1) = 0 \vee \mathrm{dst}_{t'}^\ell(c+1) = 0\right)$.

  Therefore, $\min\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right) = 0$, and $\mathrm{dst}_{t'+1}^\ell(c) = 1$.

$\square$

Theorem 6.6 (The Output Field is synchronized)**.**

$$\forall \ell t c, \mathrm{Out}_t^\ell(c) \Rightarrow \forall c', \mathrm{Out}_t^\ell(c')$$

Proof. We prove $\forall t \ell c, \mathrm{Out}_t^\ell(c) \Rightarrow \forall c', \mathrm{Out}_t^\ell(c')$ by case on t:

- If $t = 0$, let ℓ and c. By (8), $\mathrm{Out}_0^\ell(c)$ is False, so the implication holds.
- Else, $t = t' + 1$ and we prove $\forall \ell c, \mathrm{Out}_{t'+1}^\ell(c) \Rightarrow \forall c', \mathrm{Out}_{t'+1}^\ell(c')$ by induction on ℓ:
  - $\mathrm{Out}_{t'+1}^0(c)$ implies (8) $\mathrm{Brd}_{t'}^0(c-1)$ and $\mathrm{Brd}_{t'}^0(c)$ and $\mathrm{Brd}_{t'}^0(c+1)$, so (2) we have $c - 1 = 1 \vee c - 1 = n$ and $c = 1 \vee c = n$ and $c + 1 = 1 \vee c + 1 = n$, which leads to a contradiction (three variables with distinct values, but only two available values).
  - We assume the induction hypothesis:

    $$\forall c, \mathrm{Out}_{t'+1}^\ell(c) \Rightarrow \forall c', \mathrm{Out}_{t'+1}^\ell(c') \qquad (IH_\ell)$$

    Let c. The hypothesis $\mathrm{Out}_{t'+1}^{\ell+1}(c)$ implies (8) that $\mathrm{Brd}_{t'}^{\ell+1}(c-1)$ and $\mathrm{Brd}_{t'}^{\ell+1}(c)$ and $\mathrm{Brd}_{t'}^{\ell+1}(c+1)$.

    For each cell $c' \in \{c-1, c, c+1\}$, by using the lemma 6.2 we have $\mathrm{Brd}_{t'}^\ell(c') \vee \neg \mathrm{Brd}_{t'}^\ell(c')$. But because $\mathrm{Brd}_{t'}^{\ell+1}(c')$ implies (2) that $\mathrm{Brd}_{t'}^\ell(c') \vee \mathrm{Mid}_{t'}^\ell(c')$, we have two cases: $\mathrm{Brd}_{t'}^\ell(c')$ or $\neg \mathrm{Brd}_{t'}^\ell(c') \wedge \mathrm{Mid}_{t'}^\ell(c')$.

    We prove $\forall c', \mathrm{Out}_{t'+1}^{\ell+1}(c')$ for the eight possible cases:

    * If $\mathrm{Brd}_{t'}^\ell(c-1)$ and $\mathrm{Brd}_{t'}^\ell(c)$ and $\mathrm{Brd}_{t'}^\ell(c+1)$ then (8) $\mathrm{Out}_{t'+1}^\ell(c)$. So, by using IH_ℓ we have for every c' that $\mathrm{Out}_{t'+1}^\ell(c')$. Therefore, by using the lemma 6.3, we have $\mathrm{Out}_{t'+1}^{\ell+1}(c')$.
    * If $\mathrm{Mid}_{t'}^\ell(c-1)$ and $\mathrm{Mid}_{t'}^\ell(c)$ and $\mathrm{Mid}_{t'}^\ell(c+1)$, we obtain a contradiction by using the lemma 6.4.
    * The other cases are :
      · $\mathrm{Brd}_{t'}^\ell(c-1)$ and $\mathrm{Brd}_{t'}^\ell(c)$ and $\mathrm{Mid}_{t'}^\ell(c+1)$
      · $\mathrm{Brd}_{t'}^\ell(c-1)$ and $\mathrm{Mid}_{t'}^\ell(c)$ and $\mathrm{Brd}_{t'}^\ell(c+1)$
      · $\mathrm{Brd}_{t'}^\ell(c-1)$ and $\mathrm{Mid}_{t'}^\ell(c)$ and $\mathrm{Mid}_{t'}^\ell(c+1)$
      · $\mathrm{Mid}_{t'}^\ell(c-1)$ and $\mathrm{Brd}_{t'}^\ell(c)$ and $\mathrm{Brd}_{t'}^\ell(c+1)$
      · $\mathrm{Mid}_{t'}^\ell(c-1)$ and $\mathrm{Brd}_{t'}^\ell(c)$ and $\mathrm{Mid}_{t'}^\ell(c+1)$
      · $\mathrm{Mid}_{t'}^\ell(c-1)$ and $\mathrm{Mid}_{t'}^\ell(c)$ and $\mathrm{Brd}_{t'}^\ell(c+1)$

In every case, there exists a cell $m \in \{c-1, c, c+1\}$ with is a middle, not a border, and is adjacent to a border. So, by using the lemma 6.5 we have $\mathrm{dst}_{t'}^{\ell}(m) = 1$.
Let c' be a cell. By using the lemma 5.4 we have that:

$$\mathrm{dst}_{t'}^{\ell}(c') \leq \mathrm{dst}_{t'}^{\ell}(m) = 1$$

We prove that $\mathrm{Brd}_{t'}^{\ell+1}(c')$ by case on $\mathrm{dst}_{t'}^{\ell}(c')$:

· In the case $\mathrm{dst}_{t'}^{\ell}(c') = 0$, by using the lemma 5.6 we have that $\mathrm{Sta}_{t'}^{\ell}(c')$, so by using the lemma 2.7 we have that $\mathrm{Brd}_{t'}^{\ell}(c')$.

· If $\mathrm{dst}_{t'}^{\ell}(c') = 1$, by using the lemma 5.5 we have that $\mathrm{Mid}_{t'}^{\ell}(c')$.

Therefore, in every case $\mathrm{Brd}_{t'}^{\ell+1}(c')$.
We proved it for every cell c', so we have $\mathrm{Brd}_{t'}^{\ell+1}(c'-1)$ and $\mathrm{Brd}_{t'}^{\ell+1}(c')$ and $\mathrm{Brd}_{t'}^{\ell+1}(c'+1)$, therefore $\mathrm{Out}_{t'+1}^{\ell+1}(c')$.

□

7 APPENDIX

Lemma 7.1.

$$\forall n d_1 d_2, (n = 2d_1 \lor n = 2d_1 + 1) \land (n = 2d_2 \lor n = 2d_2 + 1) \Rightarrow d_1 = d_2$$

Proof. The proof is made by case on n:

- In this case, n is even.

  Because $n = 2d_1 \lor n = 2d_1 + 1$ and n is even, we have that $n = 2d_1$.

  Because $n = 2d_2 \lor n = 2d_2 + 1$ and n is even, we have that $n = 2d_2$.

  So $2d_1 = 2d_2$, therefore $d_1 = d_2$.

- In this case, n is odd.

  Because $n = 2d_1 \lor n = 2d_1 + 1$ and n is odd, we have that $n = 2d_1 + 1$.

  Because $n = 2d_2 \lor n = 2d_2 + 1$ and n is odd, we have that $n = 2d_2 + 1$.

  So $2d_1 + 1 = 2d_2 + 1$, therefore $d_1 = d_2$.

□