Proving Formally a Field-Based FSSP Solution

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Abstract

In the research domain of cellular automata algorithmics, the firing squad synchronization problem is a famous problem that has been solved in many ways. However, very few of these solutions have a detailed formal proof suitable for proof assistant and for human understanding. The reason is that the correctness of those solutions is more easily seen from a high level point of view and from the way the solution has been constructed than from the actual formal description of the constructed solution. The field-based approach consists in giving a formal description of a new high-level and modular description and the reduction process down to the final low-level solution. The decomposition in modules are natural from a design point of view and is a precise counterpart of the previous informal high-level description. This research report aims to present an understandable proof of correctness of the high-level field-based solution suitable for an implementation in Coq.

1 Introduction

This research report presents a formal proof to be implemented in Coq for a high-level solution to the multi-general one dimensional firing squad synchronization problem. The problem is to design a cellular automaton $\langle \Sigma, \delta : \Sigma^3 \to \Sigma \rangle$ with three special states $\mathtt{q}, \mathtt{g}, \mathtt{f} \in \Sigma$ called the quiescent, the general and the firing state respectively with the following synchronization property. Given a line of cells of arbitrary length with some cells in the general state and all the other cells in the quiescent states, the firing state is netherthless reached by all cells of the line at the exact same time. This has to be the case despite the fact that δ is required to have $\delta(s_1, \mathtt{q}, s_2) = \mathtt{q}$ whenever $s_1, s_2 \in \{\perp, \mathtt{q}\}$, which causes an initial asynchronous starting of the cells. Here $\perp$ is a special state representing the absence of some neighbors at each extremity of the line of cells (we discuss this further in the appendix p.20). For more information on this problem definition and variation, one can consult [**?**].

TODO: Provide a list of existing solutions having a formal proof of correctness and discuss them. In particular, discuss the Mazoyer's solutions and their proofs in Coq [**?**, **?**, **?**].

We give at the section 2 p.2 an example of evolution. A cell is labeled by a number between 1 and n. The generals are the cells awaken at the date $t = 0$, and at every date t the local informations of a cell c and its neighbors determine the next state of c at the date $t + 1$. The synchronization is obtained by recursion on the **layers**. At the layer $\ell = 0$ the borders are the cells 1 and n. At every layer ℓ the middles of the regions are computed during the evolution, and become borders for the layer $\ell + 1$. Therefore, the space is split in regions, and subregions, and so on until every cell become a border. Then, they fire together.

These local informations are formalized as different propositional **fields** at the section 3 p.4, which define the rules of the evolution:

- $\mathrm{Inp}_t(c)$ denotes that the cell c is awaken at the date t.
- $\mathrm{Brd}_t^\ell(c)$ denotes that at the layer ℓ the cell c knows at the date t that it is a border of a region.

- $\mathrm{Ins}_t^\ell(c)$ denotes that at the layer ℓ the cell c knows at the date t that it is (strictly) inside a region.
- $\mathrm{dst}_t^\ell(c)$ denotes that at the layer ℓ the cell c knows at the date t that it is at least at a distance $\geq d$ from the borders.
- $\mathrm{Sta}_t^\ell(c)$ denotes that at the layer ℓ the cell c knows at the date t that its state will not change during the following dates.
- $\mathrm{Mid}_t^\ell(c)$ denotes that at the layer ℓ the cell c knows at the date t that it is a middle of a region.
- $\mathrm{Out}_t^\ell(c)$ denotes that at the layer ℓ the cell c knows at the date t that it can fire.

Most of the lemmas and propositions highlight that the fields compute what they are supposed to. For example, we prove at the lemma B.5 p.22 that the border and inside field are exclusive, and at the lemma B.6 p.23 that a border has a distance $d = 0$.

Until every cell has enough local informations to deduce that globally every cell can fire at the the same time, the fields accumulate more and more local information over time. This is illustrated by the notion of **monotonicity** for the fields, which means that if the property holds for a given time t, then it holds for every $t' \geq t$. We prove in the appendix at the section C p.25 that for a given layer ℓ the fields are monotone.

An other core notion of the paper is the concept of **Light Cone**, formalized by the field $\mathrm{LC}_t^\ell(b_1, b_2)$ which denotes that the cells b_1 and b_2 are borders, and that at the date $t+1$ every cell (strictly) between b_1 and b_2 is inside. In a sense, this is a spatiotemporal version of the strictly spatial notion of region presented in the references. We prove in the section 5 p.11 that the Light Cones are the global counterpart of the local middles of the regions. More precisely, we prove at the lemma 5.9 p.14 that these information are necessary, and at the corollaries 5.5 p.12 and 5.6 p.13 that these informations are sufficient, to compute the middle(s) of the region.

The Light Cones help us prove in the section 6 p.15 the proposition 6.3 p.15 stating that the middles of the regions appear at the same time with the same distance, in order to prove the synchronization.

The purpose of this research report is to prove that the synchronization is effective. In our framework, the proof is done at the theorem 7.4 p.17, stating that for every layer ℓ, for every date t and every cell c, if $\mathrm{Out}_t^\ell(c)$ (ie the cell fires) then for every cell c' we have $\mathrm{Out}_t^\ell(c')$ (ie every cell fire).

Finally, we discuss some issues in the conclusion p.19. We let the most technical proofs to the appendix, and at the section F p.48 we provide a code in Coq of the definitions and some lemmas.

+ biblio dans tout l'article

2 Example

Before defining formally the fields, we introduce them in an example of execution at the table 1.

Given one or several generals, the aim is to ensure that every cell will fire at the same time.

In this example, there are seven cells. Each cell is represented by a column, and each line corresponds to a date. So, the table should be read line by line, from top to bottom.

The white cells represent the cells which are not awaken. At $t = 0$, only the fifth cell is awaken: this is the **general**. At each step, each cell **awakes** its neighbors if they are not already awaken.

cells time	1	2	3	4	5	6	7
$t = 0$					0		
$t = 1$				1	1	1	
$t = 2$			1	1	2	1	**0**
$t = 3$		1	1	2	2	1	**0**
$t = 4$	**0**	1	2	2	2	1	**0**
$t = 5$	**0**	1	2	3	2	1	**0**
$t = 6$	**0**	1	2	3	2	1	**0**
$t = 7$	**0**	1	2	**3**	2	1	**0**
$t = 8$	**0**	1	2	**3**	2	1	**0**
$t = 9$	**0**	1	2	**3**	2	1	**0**
$t = 10$	**0**	1	2	**3**	2	1	**0**

Table 1: Layer $\ell = 0$

When a cell is awaken, it begins to compute its possible **distance** to the borders. Because the evolution is given only by local rules, this computation must be done by a cell only by looking to its neighbors. If a cell is a **border**, its distance to the borders is 0, else it is $1 + \min(d_\ell, d_r)$, where d_ℓ and d_r are the distance of its left and right neighbors at the previous step. Notice that the distance in each cell is increasing over time.

The gray cells represent the cells which are able to know that they have computed the right distance to the borders. A border knows that it has a distance 0 to the borders, and a cell which has a distance $1 + d$ and a stable neighbor with a distance d becomes **stable**.

The cells form a **region**, delimited by the borders $c = 1$ and $c = 7$. The middle of this region is the cell $c = 4$. This cell knows that it is a middle when its neighbors are stable and have a lower distance to th borders. The distance of the borders and the middles are written in bold to highlight the region. After $t = 7$, the distance are stable and the middle has appeared, so nothing change.

Our aim is to divide the region in half, then divide the **subregions** in half, and so on until every cell is a border. Then, they will fire at the same time.

In order to do that, we introduce the notion of **layers**. The previous computations have been done for the layer $\ell = 0$. In the next layer ($\ell = 1$), we assume that a cell knows that it is a border when it knows that it is a border or a middle at the previous layer ($\ell = 0$).

Moreover, a cell will compute its distance to the borders only if it knows it is a border or is inside the region. In other word, a cell is awaken if and only if it is a border or inside. More precisely, a cell becomes inside at a layer $\ell > 0$ if at the previous layer it was inside and stable, and had at the previous step a neighbor with a greater distance.

Therefore, the execution at the layer $\ell = 1$ is given at the table 2.

Notice that because there is an even number of cells in the two regions of the layer $\ell = 1$, two middles appear.

cells time	1	2	3	4	5	6	7
$t=0$							
$t=1$							
$t=2$							**0**
$t=3$						1	**0**
$t=4$	**0**					1	**0**
$t=5$	**0**	1				1	**0**
$t=6$	**0**	1	1		1	1	**0**
$t=7$	**0**	1	1	**0**	1	1	**0**
$t=8$	**0**	1	1	**0**	1	1	**0**
$t=9$	**0**	**1**	**1**	**0**	**1**	**1**	**0**
$t=10$	**0**	**1**	**1**	**0**	**1**	**1**	**0**

cells time	1	2	3	4	5	6	7
$t=0$							
$t=1$							
$t=2$							**0**
$t=3$							**0**
$t=4$	**0**						**0**
$t=5$	**0**						**0**
$t=6$	**0**						**0**
$t=7$	**0**			**0**			**0**
$t=8$	**0**			**0**			**0**
$t=9$	**0**	**0**	**0**	**0**	**0**	**0**	**0**
$t=10$	`f`	`f`	`f`	`f`	`f`	`f`	`f`

Table 2: Layers $\ell = 1$ and $\ell = 2$

At the layer $\ell = 2$, every cell become a border at $t = 9$. So, at $t = 10$, each cell knows that its neighbors and itself are borders, and fires.

The number of layers is potentially infinite, but notice that every layer $\ell \geq 2$ is the same than the layer $\ell = 2$. This, and the fact that the distance fields has a maximum value depending of the total number of cells, implies that the automaton described in this paper has an infinite number of state. But it is proven in [**?**] that these states can be finitely implemented, so we will not discuss this further in this paper.

3 Fields

For the Coq implementation, the fields are computed using **booleans** but the results will be proven using **propositions** as recommanded in Software Foundations (Benjamin Pierce).

The definition are simplified (quantifier elimination) for the implementation, as opposed to the paper (cite finitization).

We assume in this paper that the problem is one-dimensional, and the cells are labeled from 1 to n.

Axiom 3.1 (At least three cells)**.**

$$n \geq 3$$

The boolean function gen is given by the evolution, and indicates the position of the **general(s)** at the beginning of the execution.

Axiom 3.2 (At least one cell is a general)**.**

$$\exists c, 1 \leq c \leq n \wedge \mathrm{gen}(c) = \mathrm{true}$$

In the following definitions, we assume that " or ", " and " and "if ...then ...else ..." are the standard booleans operations.

The **input** field indicates which cells are awaken at a particular time t. For $t = 0$, only the generals are awaken, and at each step an awaken cell awakes its neighboring cells:

Definition 3.3 (Input Field)**.**

$$\begin{aligned}
\mathrm{inp}_0(c) &\stackrel{\mathrm{def}}{=} \mathrm{gen}(c) \\
\mathrm{inp}_{t+1}(c) &\stackrel{\mathrm{def}}{=} \mathrm{inp}_t(c-1) \text{ or } \mathrm{inp}_t(c) \text{ or } \mathrm{inp}_t(c+1)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Inp}_0(c) &\stackrel{\mathrm{def}}{=} \mathrm{gen}(c) = \mathrm{true} \\
\mathrm{Inp}_{t+1}(c) &\stackrel{\mathrm{def}}{=} \mathrm{Inp}_t(c-1) \vee \mathrm{Inp}_t(c) \vee \mathrm{Inp}_t(c+1) \qquad (1)
\end{aligned}$$

Like this definition, the boolean fields will be written with lowercase, and the proposition fields in uppercase. The Coq file at the section F p.48 contains already the proof of equivalence, so they will be admitted in this report.

Lemma 3.4 (Equivalence for Inp)**.**

$$\forall tc, \mathrm{Inp}_t(c) \Leftrightarrow \mathrm{inp}_t(c) = \mathrm{true}$$

For the sake of clarity, the $=$ and $<$ will not be distinguished from their boolean equivalent, as it is in Coq. The recursive definition of the proposition fields is given at the table 3, where dst is an integer field computed along the booleans fields.

Remark. Notice that if $c = 1$ then the cell $c - 1$ is not in our space, neither $c + 1$ for $c = n$.

Therefore, the definition of the fields should be modified as in [**?**] to take a proper neighborhood into account.

We discuss in the appendix p.20 what is required to be modified in the formal definition of the fields for the cells $c \in \{1, n\}$, but we keep here the non-modified version because it is simpler and it corresponds to our implementation in Coq p.48.

Coq cannot guess how to compute such an intricated recursion, so the recursive definitions of the booleans fields must be split into abstract parts for different given levels:

1. We use the input field to define at the table 4 the **border** and **inside** fields at the level 0.
2. Then, we assume that the border and inside fields are defined at the level ℓ, and we define at the table 5 the **distance**, **stability** and **middle** fields at that level.
3. Finally, we use the fields defined at the level ℓ to define at the table 6 the border and inside fields for the level $\ell + 1$.

The boolean fields should be defined by the mutual recursion given at the table 7, but Coq cannot guess the decreasing argument.

So, instead, we substitute the schemata at the table 8 to obtain only one mutual recursion for the border and inside fields and thereafter define the other fields, where $f(g_1, \ldots, g_k)$ denotes the field $(t, c) \mapsto f(t, c, g_1, \ldots, g_k)$.

And this time, Coq is able to compute the fields. Moreover we prove in Coq at the section F p.48 the equivalence between the boolean and proposition fields, and the specification of dst:

$$
\begin{aligned}
\mathrm{Brd}^0_t(c) &\stackrel{\text{def}}{=} \mathrm{Inp}_t(c) \wedge (1 = c \vee c = n) \\
\mathrm{Brd}^{\ell+1}_t(c) &\stackrel{\text{def}}{=} \mathrm{Brd}^\ell_t(c) \vee \mathrm{Mid}^\ell_t(c) \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ins}^0_t(c) &\stackrel{\text{def}}{=} \mathrm{Inp}_t(c) \wedge 1 < c \wedge c < n \\
\mathrm{Ins}^{\ell+1}_0(c) &\stackrel{\text{def}}{=} \mathrm{False} \\
\mathrm{Ins}^{\ell+1}_{t+1}(c) &\stackrel{\text{def}}{=} \mathrm{Ins}^\ell_{t+1}(c) \wedge \mathrm{Sta}^\ell_{t+1}(c) \\
&\quad \wedge \Big(\mathrm{dst}^\ell_{t+1}(c) < \mathrm{dst}^\ell_t(c-1) \vee \mathrm{dst}^\ell_{t+1}(c) < \mathrm{dst}^\ell_t(c+1)\Big) \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Sta}^\ell_0(c) &\stackrel{\text{def}}{=} \mathrm{Brd}^\ell_0(c) \\
\mathrm{Sta}^\ell_{t+1}(c) &\stackrel{\text{def}}{=} \mathrm{Brd}^\ell_{t+1}(c) \\
&\quad \vee \Big(\mathrm{dst}^\ell_{t+1}(c) = 1 + \mathrm{dst}^\ell_t(c-1) \wedge \mathrm{Sta}^\ell_t(c-1)\Big) \\
&\quad \vee \Big(\mathrm{dst}^\ell_{t+1}(c) = 1 + \mathrm{dst}^\ell_t(c+1) \wedge \mathrm{Sta}^\ell_t(c+1)\Big) \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Mid}^\ell_0(c) &\stackrel{\text{def}}{=} \mathrm{False} \\
\mathrm{Mid}^\ell_{t+1}(c) &\stackrel{\text{def}}{=} \Bigg(\mathrm{dst}^\ell_{t+1}(c) > \max\Big(\mathrm{dst}^\ell_t(c-1), \mathrm{dst}^\ell_t(c+1)\Big) \\
&\qquad \wedge \mathrm{Sta}^\ell_t(c-1) \wedge \mathrm{Sta}^\ell_t(c+1)\Bigg) \\
&\quad \vee \Bigg(\mathrm{dst}^\ell_{t+1}(c) = \max\Big(\mathrm{dst}^\ell_t(c-1), \mathrm{dst}^\ell_t(c+1)\Big) \\
&\qquad \wedge \mathrm{Sta}^\ell_t(c-1) \wedge \mathrm{Sta}^\ell_t(c) \wedge \mathrm{Sta}^\ell_t(c+1)\Bigg) \qquad (5)
\end{aligned}
$$

Table 3: Formal Definition of the Fields

$$\mathrm{brd0}(t, c) \stackrel{\text{def}}{=} \mathrm{inp}_t(c) \ \text{ and } \ (1 = c \ \text{ or } \ c = n)$$

$$\mathrm{ins0}(t, c) \stackrel{\text{def}}{=} \mathrm{inp}_t(c) \ \text{ and } \ 1 < c \ \text{ and } \ c < n$$

Table 4: Border and Inside Fields at the Layer 0

$$
\begin{aligned}
\mathrm{dstL}(0, c, \mathrm{insL}) &\stackrel{\text{def}}{=} 0 \\
\mathrm{dstL}(t+1, c, \mathrm{insL}) &\stackrel{\text{def}}{=} \text{if } \mathrm{insL}(t+1, c) \\
&\quad\ \text{then } 1 + \min\Big(\mathrm{dstL}(t, c-1), \mathrm{dstL}(t, c+1)\Big) \\
&\quad\ \text{else } 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{staL}(0, c, \mathrm{brdL}, \mathrm{dstL}) &\stackrel{\text{def}}{=} \mathrm{brdL}(0, c) \\
\mathrm{staL}(t+1, c, \mathrm{brdL}, \mathrm{dstL}) &\stackrel{\text{def}}{=} \mathrm{brdL}(t+1, c) \\
&\quad\ \text{or } \Big(\mathrm{dstL}(t+1, c) = 1 + \mathrm{dstL}(t, c-1) \text{ and } \mathrm{staL}(t, c-1)\Big) \\
&\quad\ \text{or } \Big(\mathrm{dstL}(t+1, c) = 1 + \mathrm{dstL}(t, c+1) \text{ and } \mathrm{staL}(t, c+1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{midL}(0, c, \mathrm{dstL}, \mathrm{staL}) &\stackrel{\text{def}}{=} \text{false} \\
\mathrm{midL}(t+1, c, \mathrm{dstL}, \mathrm{staL}) &\stackrel{\text{def}}{=} \Bigg(\mathrm{dstL}(t+1, c) > \max\Big(\mathrm{dstL}(t, c-1), \mathrm{dstL}(t, c+1)\Big) \\
&\qquad \text{and } \mathrm{staL}(t, c-1) \text{ and } \mathrm{staL}(t, c+1)\Bigg) \\
&\quad\ \text{or } \Bigg(\mathrm{dstL}(t+1, c) = \max\Big(\mathrm{dstL}(t, c-1), \mathrm{dstL}(t, c+1)\Big) \\
&\qquad \text{and } \mathrm{staL}(t, c-1) \text{ and } \mathrm{staL}(t, c) \text{ and } \mathrm{staL}(t, c+1)\Bigg)
\end{aligned}
$$

Table 5: Distance, Stability and Middle Fields at the Layer ℓ

$$
\mathrm{brdS}(t, c, \mathrm{brdL}, \mathrm{midL}) \stackrel{\text{def}}{=} \mathrm{brdL}(t, c) \text{ or } \mathrm{midL}(t, c)
$$

$$
\begin{aligned}
\mathrm{insS}(0, c, \mathrm{insL}, \mathrm{dstL}, \mathrm{staL}) &\stackrel{\text{def}}{=} \text{false} \\
\mathrm{insS}(t+1, c, \mathrm{insL}, \mathrm{dstL}, \mathrm{staL}) &\stackrel{\text{def}}{=} \mathrm{insL}(t+1, c) \text{ and } \mathrm{staL}(t+1, c) \\
&\qquad \text{and } \Big(\mathrm{dstL}(t+1, c) < \mathrm{dstL}(t, c-1) \\
&\qquad\qquad \text{or } \mathrm{dstL}(t+1, c) < \mathrm{dstL}(t, c-1)\Big)
\end{aligned}
$$

Table 6: Border and Inside Fields at level $\ell + 1$

Lemma 3.5 (Equivalence between Boolean and Proposition Fields)**.**

$$
\begin{gathered}
\forall \ell t c, \mathrm{Brd}_t^\ell(c) \Leftrightarrow \mathrm{brd}_t^\ell(c) = \text{true} \\
\forall \ell t c, \mathrm{Ins}_t^\ell(c) \Leftrightarrow \mathrm{ins}_t^\ell(c) = \text{true} \\
\forall \ell t c, \mathrm{Sta}_t^\ell(c) \Leftrightarrow \mathrm{sta}_t^\ell(c) = \text{true} \\
\forall \ell t c, \mathrm{Mid}_t^\ell(c) \Leftrightarrow \mathrm{mid}_t^\ell(c) = \text{true}
\end{gathered}
$$

$$
\begin{aligned}
\mathrm{brd}^0_t(c) &\stackrel{\text{def}}{=} \mathrm{brd0}(t,c)\\
\mathrm{brd}^{\ell+1}_t(c) &\stackrel{\text{def}}{=} \mathrm{brdS}(t,c,\mathrm{brd}^\ell,\mathrm{mid}^\ell)\\[1em]
\mathrm{ins}^0_t(c) &\stackrel{\text{def}}{=} \mathrm{ins0}(t,c)\\
\mathrm{ins}^{\ell+1}_t(c) &\stackrel{\text{def}}{=} \mathrm{insS}(t,c,\mathrm{ins}^\ell,\mathrm{dst}^\ell,\mathrm{sta}^\ell)\\[1em]
\mathrm{dst}^\ell_t(c) &\stackrel{\text{def}}{=} \mathrm{dstL}(t,c,\mathrm{ins}^\ell)\\[1em]
\mathrm{sta}^\ell_t(c) &\stackrel{\text{def}}{=} \mathrm{staL}(t,c,\mathrm{brd}^\ell,\mathrm{dst}^\ell)\\[1em]
\mathrm{mid}^\ell_t(c) &\stackrel{\text{def}}{=} \mathrm{midL}(t,c,\mathrm{dst}^\ell,\mathrm{sta}^\ell)
\end{aligned}
$$

Table 7: Mutual Recursion for the Boolean Fields (first try)

$$
\begin{aligned}
\mathrm{brd}^0 &\stackrel{\text{def}}{=} \mathrm{brd0}\\
\mathrm{brd}^{\ell+1} &\stackrel{\text{def}}{=} \mathrm{brdS}\Bigg(\mathrm{brd}^\ell,\mathrm{midL}\bigg(\mathrm{dstL}\Big(\mathrm{ins}^\ell\Big),\mathrm{staL}\Big(\mathrm{brd}^\ell,\mathrm{dstL}\big(\mathrm{ins}^\ell\big)\Big)\bigg)\Bigg)\\[1em]
\mathrm{ins}^0 &\stackrel{\text{def}}{=} \mathrm{ins0}\\
\mathrm{ins}^{\ell+1} &\stackrel{\text{def}}{=} \mathrm{insS}\Big(\mathrm{ins}^\ell,\mathrm{dstL}\Big(\mathrm{ins}^\ell\Big),\mathrm{staL}\Big(\mathrm{brd}^\ell,\mathrm{dstL}\big(\mathrm{ins}^\ell\big)\Big)\Big)\\[1em]
\mathrm{dst}^\ell &\stackrel{\text{def}}{=} \mathrm{dstL}(\mathrm{ins}^\ell)\\[1em]
\mathrm{sta}^\ell &\stackrel{\text{def}}{=} \mathrm{staL}(\mathrm{brd}^\ell,\mathrm{dst}^\ell)\\[1em]
\mathrm{mid}^\ell &\stackrel{\text{def}}{=} \mathrm{midL}(\mathrm{dst}^\ell,\mathrm{sta}^\ell)
\end{aligned}
$$

Table 8: Mutual Recursion for the Boolean Fields (second try)

Lemma 3.6 (Distance Field)**.**

$$
\begin{aligned}
\mathrm{dst}^\ell_0(c) &= 0\\
\mathrm{Ins}^\ell_{t+1}(c) \Rightarrow \mathrm{dst}^\ell_{t+1}(c) &= 1+\min\Big(\mathrm{dst}^\ell_t(c-1),\mathrm{dst}^\ell_t(c+1)\Big)\\
\neg\,\mathrm{Ins}^\ell_{t+1}(c) \Rightarrow \mathrm{dst}^\ell_{t+1}(c) &= 0 \qquad (6)
\end{aligned}
$$

In the rest of the paper, we will use only the axioms 3.1 p.4 (there exists at least three cells) and 3.2 p.4 (there exists at least one general), and the equations (1) for the input field, (2) for the border field, (3) for the inside field, (4) for the stable field, (5) for the middle field, and (6) for the distance field.

4 Framework

In the appendix at the section B p.20 we detail technical lemmas about the fields. Their proofs are very detailed and written in a Coq style, in order to be implemented. In fact, some (but not all) have been at the section F p.48.

We prove at the lemma B.5 p.22 that the border and inside fields are exclusive, which means more formally that:

$$\forall \ell t c, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{False}$$

We prove at the lemma B.7 p.23 that the neighbors of a middle are stable, and at the lemma B.8 p.23 that a middle itself is stable too:

$$\forall \ell t c, \mathrm{Mid}_{t+1}^\ell(c) \Rightarrow \mathrm{Sta}_t^\ell(c-1) \wedge \mathrm{Sta}_t^\ell(c+1)$$

$$\forall \ell t c, \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{Sta}_t^\ell(c)$$

By the stable field equation (4), a border is stable. Therefore, by using the lemmas B.6 p.23 and B.9 p.24 we have the following equivalence:

$$\forall \ell t c, \mathrm{Brd}_t^\ell(c) \Leftrightarrow \mathrm{Sta}_t^\ell(c) \wedge \mathrm{dst}_t^\ell(c) = 0$$

Finally, the lemma B.11 p.24 states that at the layer 0, every cell ends up being awaken:

$$\exists t, \forall c, \mathrm{Inp}_t(c)$$

At the section C p.25 of the appendix, we prove the **monotonicity** of the fields, which means that if the property holds for a given time t, then it holds for every $t' \geq t$. As for the section B, The proof are very detailed, and some (but not all) have been implemented in Coq at the section F p.48.

The proof of the monotonicity of the input field derives directly from the input field equation (1), so we have the lemma C.1 p.25:

$$\forall t c, \mathrm{Inp}_t(c) \Rightarrow (\forall t', t' \geq t \Rightarrow \mathrm{Inp}_{t'}(c))$$

The other fields are defined for each layer by mutual recursion, so the proof must be split into several parts.

Firstly, we prove the monotonicity of the border and inside fields at the layer $\ell = 0$.

Secondly, we prove that if the border and inside fields are monotone at the layer ℓ, then the other fields are monotone at the layer ℓ too and, moreover, that the border and inside fields are monotone at the layer $\ell + 1$.

1. For the layer $\ell = 0$, if $\mathrm{Brd}_t^0(c)$ then according to the border field equation (2), we have $\mathrm{Inp}_t(c)$ and $c = 1$ or n. But, according to the lemma C.1, the input field is monotone, so we have $\mathrm{Inp}_{t+1}(c)$ and $c = 1$ or n. Therefore, we have $\mathrm{Brd}_{t+1}^0(c)$.

   In the same way, if $\mathrm{Ins}_t^0(c)$ then according to the border field equation (3), we have $\mathrm{Inp}_t(c)$ and $1 < c < n$. But, according to the lemma C.1, the input field is monotone, so we have $\mathrm{Inp}_{t+1}(c)$ and $1 < c < n$. Therefore, we have $\mathrm{Ins}_{t+1}^0(c)$.

2. Let ℓ be a layer. We assume that the border and inside fields are monotone at the layer ℓ:

$$\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c)$$

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)$$

Then, we prove in the lemma C.2 p.25 that the distance field is increasing at the layer ℓ:

$$\forall tc, \mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_{t+1}^\ell(c)$$

Then, we prove at the lemma C.3 p.25 that a cell which is stable at the layer ℓ has a constant distance:

$$\forall tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) = \mathrm{dst}_{t+1}^\ell(c)$$

Then, we prove at the lemma C.4 p.26 that the stable field is monotone at the layer ℓ:

$$\forall tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{Sta}_{t+1}^\ell(c)$$

Then, we prove at the lemma C.5 p.28 that the middle field is monotone at the layer ℓ:

$$\forall tc, \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{Mid}_{t+1}^\ell(c)$$

Then, we prove at the lemma C.6 p.29 that the border field is monotone at the layer $\ell+1$:

$$\forall tc, \mathrm{Brd}_t^{\ell+1}(c) \Rightarrow \mathrm{Brd}_{t+1}^{\ell+1}(c)$$

Then, we prove at the lemma C.7 p.30 that the border field is monotone at the layer $\ell+1$:

$$\forall tc, \mathrm{Ins}_t^{\ell+1}(c) \Rightarrow \mathrm{Ins}_{t+1}^{\ell+1}(c)$$

Therefore, we proved that the fields are monotone at every layer:

$$\forall \ell tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Brd}_{t'}^\ell(c)\Big)$$

$$\forall \ell tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Ins}_{t'}^\ell(c)\Big)$$

$$\forall \ell tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Sta}_{t'}^\ell(c)\Big)$$

$$\forall \ell tc, \mathrm{Mid}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Mid}_{t'}^\ell(c)\Big)$$

$$\forall \ell tct', t' \geq t \Rightarrow \mathrm{dst}_{t'}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$$

$$\forall \ell tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{dst}_{t'}^\ell(c) = \mathrm{dst}_t^\ell(c)\Big)$$

cells / time	1	2	3	4	5	6	7
$t = 0$	**0**						**0**
$t = 1$	**0**	1				1	**0**
$t = 2$	**0**	1	1		1	1	**0**
$t = 3$	**0**	1	1	2	1	1	**0**
$t = 4$	**0**	1	2	2	2	1	**0**
$t = 5$	**0**	1	2	**3**	2	1	**0**

Table 9: A light cone for 7 cells.

5 Light Cones

What are the necessary informations to produce a middle ?

The information travels from cell to cell at the speed of one cell per step, which is in a way the light speed for the automaton. So, the necessary informations required to produce a middle must come from the borders, travel from cell to cell in diagonal, and attain the middle(s) at the step it appears.

At the table 9, we give an example of a "worst" case, where the middle of the automaton has little to no information during most of the execution, because the generals are at the cells 1 and 7. The middle appears at the step $t = 5$, and then the evolution does not change anymore. The informations leading to the middle at $t = 5$ traveled from the entire region at $t = 2$, which we call the **Light Cone** of the middle.

Notice that some cells may not be awaken at $t = 2$, but they are all border or inside after the first step of the Light Cone.

So, in this example it is true that at the layer $\ell = 0$ and the date $t = 2$ the region between the borders 1 and 7 is a Light Cone for the middle to come. This will be denoted by $\mathrm{LC}_2^0(1, 7)$ in the following definition:

Definition 5.1 (Light Cones)**.**

$$\begin{aligned} \mathrm{LC}_t^\ell(b_1, b_2) \overset{\mathrm{def}}{=} & \; b_1 + 2 \leq b_2 \wedge \mathrm{Brd}_t^\ell(b_1) \wedge \mathrm{Brd}_t^\ell(b_2) \\ & \wedge \Big(\forall c, b_1 < c < b_2 \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\Big) \end{aligned} \tag{7}$$

The condition $b_1 + 2 \leq b_2$ ensures not only that $b_1 < b_2$, but also that there is a cell between them. This ensures that the boundaries between light cones are not light cones themselves.

Moreover, this excludes the regions of the final layer (which contains only two cells) to be called light cones, so the results of this section are only for the phase transition. And indeed, the final layer is the first during the execution when a region alone cannot determine the middle(s), because a middle requires at least three cells to appear, and not only two.

Corollary 5.2 (Light Cone at layer 0)**.**

$$\exists t, \mathrm{LC}_t^0(1, n)$$

Proof. Firstly, by axiom 3.1, $n \geq 3$.

Secondly, by using the lemma B.11 there exists t such that for every cell c, $\mathrm{Inp}_t(c)$. So:

- We have (2) that $\mathrm{Brd}^0_t(1)$ and $\mathrm{Brd}^0_t(n)$
- We have (3) for every $1 < c < n$ that $\mathrm{Ins}^0_t(c)$. So, by using the corollary C.10, for every $1 < c < n$ we have that $\mathrm{Ins}^0_{t+1}(c)$.

Therefore (7) $\mathrm{LC}^0_t(1, n)$. □

Remark. In the following, $\frac{a}{2}$ will denote the floor function of the **half** : the half of a if a is even, and the half of $a - 1$ if a is odd.

As in the section 4, the proofs of the following results are very detailed and written in a Coq style, in order to be implemented. Therefore, for sake of clarity, we will only write and comment the results in this section, and leave the non-trivial proofs to the appendix at the section D p.33.

At the table 9, notice that the cells in yellow are stable, and that the distance increases from the border with $d = 0$ to the middle with $d = \frac{b_2 - b_1}{2}$. Moreover, if the Light Cone began at the date t in the borders, then at the date $t + d$ every cell inside the Light Cone has a distance $\geq d$:

Proposition 5.3 (Running of a Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^\ell_t(b_1, b_2) \Rightarrow \forall\ 0 \leq d \leq \frac{b_2 - b_1}{2},$$

$$\mathrm{dst}^\ell_{t+d}(b_1 + d) = d \wedge \mathrm{Sta}^\ell_{t+d}(b_1 + d)$$

$$\wedge\, \mathrm{dst}^\ell_{t+d}(b_2 - d) = d \wedge \mathrm{Sta}^\ell_{t+d}(b_2 - d)$$

$$\wedge \left(\forall\ b_1 + d \leq c \leq b_2 - d, \mathrm{dst}^\ell_{t+d}(c) \geq d\right)$$

Because we proved at the corollary C.11 p.32 that the stable field is monotone, and at the corollary C.14 p.32 that a cell which is stable has a constant distance over time, we can deduce from the previous proposition the state of the entire region when the middle appears at $t + \frac{b_2 - b_1}{2}$:

Corollary 5.4 (End of a Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^\ell_t(b_1, b_2) \Rightarrow \forall\ 0 \leq d \leq \frac{b_2 - b_1}{2},$$

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(b_1 + d) = d \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(b_1 + d)$$

$$\wedge\, \mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(b_2 - d) = d \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(b_2 - d)$$

Remark. Notice that for a Light Cone $\mathrm{LC}^\ell_t(b_1, b_2)$, $b_2 - b_1 + 1$ is the number of cells forming the Light Cone, boundaries included.

In our example at the table 9, the region has $7 - 1 + 1 = 7$ an odd number of cells, so one middle appeared.

Corollary 5.5 (Middle of an odd Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^\ell_t(b_1, b_2) \wedge b_2 - b_1 + 1\, \text{odd}$$

$$\Rightarrow \mathrm{Mid}^\ell_{t+\frac{b_2-b_1}{2}}\left(\frac{b_1 + b_2}{2}\right)$$

But remember that in our example at the table 2, there was an even number of cells in the two regions of the layer $\ell = 1$, so two middles appeared:

Corollary 5.6 (Middles of an even Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}_t^\ell(b_1, b_2) \wedge b_2 - b_1 + 1 \,\mathrm{even}$$

$$\Rightarrow \mathrm{Mid}^\ell_{t+\frac{b_2-b_1+1}{2}}\left(\frac{b_1+b_2-1}{2}\right) \wedge \mathrm{Mid}^\ell_{t+\frac{b_2-b_1+1}{2}}\left(\frac{b_1+b_2+1}{2}\right)$$

Remark. Notice that in every case, we have $\mathrm{Mid}^\ell_{t+\frac{b_2-b_1+1}{2}}\left(\frac{b_1+b_2}{2}\right)$ and $\mathrm{Mid}^\ell_{t+\frac{b_2-b_1+1}{2}}\left(\frac{b_1+b_2+1}{2}\right)$, but we thought the presentation clearer by separating both cases.

The main purpose of the concept of Light Cone is to help proving results about middles at the section 6 p.15. As an example, in order to prove results like the proposition 6.3 p.15, we prove that the other cells of a Light Cone are not middles:

Lemma 5.7 (The other cells of a Light Cone are not Middles)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}_t^\ell(b_1, b_2) \Rightarrow \forall t' \geq t + \frac{b_2 - b_1}{2}, \forall c,$$

$$\left(b_1 \leq c < \frac{b_1+b_2}{2} \vee \frac{b_1+b_2+1}{2} < c \leq b_2\right) \Rightarrow \neg\,\mathrm{Mid}^\ell_{t'+1}(c)$$

The previous lemma can be generalized by using the monotonicity of the middle field:

Corollary 5.8 (The other cells of a Light Cone are not Middles)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}_t^\ell(b_1, b_2) \Rightarrow \forall t' c,$$

$$\left(b_1 \leq c < \frac{b_1+b_2}{2} \vee \frac{b_1+b_2+1}{2} < c \leq b_2\right) \Rightarrow \neg\,\mathrm{Mid}^\ell_{t'}(c)$$

Proof. The proof is made by case on t':

- If $t' \leq t + \frac{b_2-b_1}{2}$, we prove $\neg\,\mathrm{Mid}^\ell_{t'}(c)$ by contradiction.

  We assume that $\mathrm{Mid}^\ell_{t'}(c)$. So, by using the lemma C.12 we have that $\mathrm{Mid}^\ell_{t+\frac{b_2-b_1}{2}+1}(c)$.

  But, by using the previous lemma with $t + \frac{b_2-b_1}{2}$, we have that $\neg\,\mathrm{Mid}^\ell_{t+\frac{b_2-b_1}{2}+1}(c)$, hence the contradiction.

- If $t' \geq t + \frac{b_2-b_1}{2} + 1$

  By using the previous lemma with $t' - 1 \geq t + \frac{b_2-b_1}{2}$, we have that $\neg\,\mathrm{Mid}^\ell_{t'}(c)$.

□

In the following, we call a **true middle** a cell which is a middle but not a border. Indeed, as opposed to the border and inside fields, these fields are not exclusive. But a cell can only be a middle and a border during the final layer, so the two fields are exclusive during the transition phase.

In a sense, the following lemma is the converse of the corollaries 5.5 and 5.6:

Lemma 5.9 (Each true Middle comes from a Light Cone).

$$\begin{gathered}\forall \ell t m, \neg \operatorname{Brd}_t^\ell(m) \wedge \operatorname{Mid}_t^\ell(m) \\ \Rightarrow \operatorname{LC}_{t-d}^\ell(m-d, m+d) \\ \vee \operatorname{LC}_{t-(d+1)}^\ell(m-(d+1), m+d) \\ \vee \operatorname{LC}_{t-(d+1)}^\ell(m-d, m+(d+1)) \\ \textit{where } d = \operatorname{dst}_t^\ell(m)\end{gathered}$$

We use the previous lemma to prove that if a (true) middle appears at the layer ℓ and the date t, then it determines the apparition of a Light Cone at the layer $\ell+1$ and at the same date:

Corollary 5.10 (A Middle induces a new Light Cone).

$$\begin{gathered}\forall \ell t m d, \operatorname{Mid}_t^\ell(m) \wedge \operatorname{dst}_t^\ell(m) = d \wedge d \geq 2 \\ \Rightarrow \forall t', \left(\operatorname{Brd}_{t'}^\ell(m-d) \Rightarrow \operatorname{LC}_t^{\ell+1}(m-d, m)\right) \\ \wedge \left(\operatorname{Brd}_{t'}^\ell(m+d) \Rightarrow \operatorname{LC}_t^{\ell+1}(m, m+d)\right)\end{gathered}$$

This allows us to prove the following proposition stating that a Light Cone at a layer ℓ will be split in half by the middle field, hence determining the formation of two Light Cones at the layer $\ell+1$:

Proposition 5.11 (A Light Cone is split in the Middle).

$$\begin{gathered}\forall \ell t b_1 b_2, \operatorname{LC}_t^\ell(b_1, b_2) \wedge b_2 - b_1 + 1 \geq 5 \\ \Rightarrow \operatorname{LC}_{t+\frac{b_2-b_1+1}{2}}^{\ell+1}\left(b_1, \frac{b_1+b_2}{2}\right) \wedge \operatorname{LC}_{t+\frac{b_2-b_1+1}{2}}^{\ell+1}\left(\frac{b_1+b_2+1}{2}, b_2\right)\end{gathered}$$

Proof. Let ℓ, t, b_1 and b_2 such that $\operatorname{LC}_t^\ell(b_1, b_2)$ and $b_2 - b_1 + 1 \geq 5$.

According to the corollary 5.5 p.12 (if b_2-b_1+1 is odd) or the corolloray 5.6 p.13 (if b_2-b_1+1 is even), we have $\operatorname{Mid}_{t+\frac{b_2-b_1+1}{2}}^\ell\left(\frac{b_1+b_2}{2}\right)$ and $\operatorname{Mid}_{t+\frac{b_2-b_1+1}{2}}^\ell\left(\frac{b_1+b_2+1}{2}\right)$.

Moreover, according to the corollary 5.4 p.12 for $d = \frac{b_2-b_1}{2}$ (and eventually the corollary C.14 p.32), we have:

$$\operatorname{dst}_{t+\frac{b_2-b_1+1}{2}}^\ell\left(\frac{b_1+b_2}{2}\right) = \frac{b_2-b_1}{2} = \operatorname{dst}_{t+\frac{b_2-b_1+1}{2}}^\ell\left(\frac{b_1+b_2+1}{2}\right)$$

Notice that the hypothesis $b_2 - b_1 + 1 \geq 5$ implies that $\frac{b_2-b_1}{2} \geq 2$.

So, because $\operatorname{LC}_t^\ell(b_1, b_2)$ implies that $\operatorname{Brd}_t^\ell(b_1)$ and $\operatorname{Brd}_t^\ell(b_2)$, we can apply the corollary 5.10 on the middle(s) to prove the result. $\square$

Finally, to help proving the proposition 6.3 p.15, we conclude this section with a lemma stating that a Light Cone at the layer $\ell+1$ with borders b_1 and b_2 determines at the layer ℓ that one was a border, and the other was a true middle:

Lemma 5.12 (One Brd and one Mid at the previous layer of a Light Cone).

$$\begin{gathered}\forall \ell t b_1 b_2, \operatorname{LC}_t^{\ell+1}(b_1, b_2) \\ \Rightarrow \left(\operatorname{Brd}_t^\ell(b_1) \wedge \neg \operatorname{Brd}_t^\ell(b_2) \wedge \operatorname{Mid}_t^\ell(b_2) \wedge \operatorname{dst}_t^\ell(b_2) = b_2 - b_1\right) \\ \vee \left(\neg \operatorname{Brd}_t^\ell(b_1) \wedge \operatorname{Mid}_t^\ell(b_1) \wedge \operatorname{Brd}_t^\ell(b_2) \wedge \operatorname{dst}_t^\ell(b_1) = b_2 - b_1\right)\end{gathered}$$

6 Middles

As in in the previous section, the proofs of the following results are very detailed and written in a Coq style, in order to be implemented. Therefore, for sake of clarity, we will only write and comment the results in this section, and leave the non-trivial proofs to the appendix at the section E p.42.

The aim of this section is to prove the proposition 6.3, which is necessary to ensure the **synchronization** of the cells (theorem 7.4 p.17). In order to alleviate the demonstration, we first prove two technical lemmas:

Lemma 6.1 (Paired Middles appear at the same time with the same distance)**.**

$$\forall \ell t_1 t_2 m_1 m_2, \mathrm{Mid}^{\ell}_{t_1}(m_1) \wedge \mathrm{Mid}^{\ell}_{t_2}(m_2) \wedge (m_2 = m_1 + 1 \vee m_1 = m_2 + 1)$$

$$\Rightarrow \mathrm{Mid}^{\ell}_{t_1}(m_2) \wedge \mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)$$

Proof. The proof is made p.43 by using the monotonicity of the fields. □

Lemma 6.2 (A Middle has the same distance over time)**.**

$$\forall \ell t_1 t_2 m, \mathrm{Mid}^{\ell}_{t_1}(m) \wedge \mathrm{Mid}^{\ell}_{t_2}(m) \Rightarrow \mathrm{dst}^{\ell}_{t_1}(m) = \mathrm{dst}^{\ell}_{t_2}(m)$$

Proof. Two cases $t_1 \leq t_2$ and $t_2 \leq t_1$. In every case, a middle is stable, therefore the distance is the same. □

Proposition 6.3 (Middles appear at the same time with the same distance)**.**

$$\forall \ell t_1 m_1, \neg \mathrm{Brd}^{\ell}_{t_1}(m_1) \wedge \mathrm{Mid}^{\ell}_{t_1}(m_1)$$

$$\Rightarrow \Big(\forall t_2 m_2, \mathrm{Mid}^{\ell}_{t_2}(m_2) \Rightarrow \mathrm{Mid}^{\ell}_{t_1}(m_2) \wedge \mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)\Big)$$

Proof. The proposition is proven p.44, but we sketch the proof here to highlight how the Light Cones are used. The proof is made by induction on the layer ℓ:

- At the layer $\ell = 0$, according to the lemma 5.2, there exists t_{LC} such that $\mathrm{LC}^0_{t_{\mathrm{LC}}}(1, n)$.

  If n is odd, then according to the corollary 5.5 we have $\mathrm{Mid}^0_{t_{\mathrm{LC}}+\frac{n-1}{2}}(\frac{n+1}{2})$.

  If n is even, then according to the corollary 5.6 we have $\mathrm{Mid}^0_{t_{\mathrm{LC}}+\frac{n}{2}}(\frac{n}{2})$ and $\mathrm{Mid}^0_{t_{\mathrm{LC}}+\frac{n}{2}}(\frac{n}{2}+1)$.

  Moreover, according to the lemma 5.8, the other cells cannot be middles.

  In the first case and potentially in the second case, we have $m_1 = m_2$, which concludes the proof.

  In the second case, if $m_1 \neq m_2$ then $m_2 = m_1 + 1$ or $m_1 = m_2 + 1$, so according to the lemma 6.1 we have $\mathrm{Mid}^0_{t_1}(m_2)$ and $\mathrm{dst}^0_{t_1}(m_1) = \mathrm{dst}^0_{t_1}(m_2)$.

- We assume the induction hypothesis for ℓ and prove it for $\ell + 1$.

  Let $d_1 = \mathrm{dst}^{\ell+1}_{t_1}(m_1)$. Because m_1 is a true middle at the layer $\ell + 1$, according to the lemma 5.9, there exists t'_1, b_1, and b'_1 such that $\mathrm{LC}^{\ell+1}_{t'_1}(b_1, b'_1)$ and $b'_1 - b_1 = 2d_1$ or $2d_1 + 1$.

  So, according to the lemma 5.12, at the layer ℓ, among b_1, and b'_1 one is a border and the other is a true middle with distance $b'_1 - b_1$. Let b^{ℓ}_1 be the border and m^{ℓ}_1 be the middle.

Let $d_2 = \mathrm{dst}_{t_2}^{\ell+1}(m_2)$. In the same way, we have $\mathrm{LC}_{t'_2}^{\ell+1}(b_2, b'_2)$, and at the previous layer the border b_2^ℓ and the middle m_2^ℓ such that $\mathrm{dst}_{t'_2}^\ell(m_2^\ell) = b'_2 - b_2 = 2d_2$ or $2d_2 + 1$.

So, according to the induction hypothesis, we have $\mathrm{Mid}_{t'_1}^\ell(m_2^\ell)$ and $\mathrm{dst}_{t'_1}^\ell(m_1^\ell) = \mathrm{dst}_{t'_1}^\ell(m_2^\ell)$.

Moreover, according to the lemma 6.2 we have $\mathrm{dst}_{t'_1}^\ell(m_2^\ell) = \mathrm{dst}_{t'_2}^\ell(m_2^\ell)$.

So $(2d_1$ or $2d_1 + 1) = \mathrm{dst}_{t'_1}^\ell(m_1^\ell) = \mathrm{dst}_{t'_1}^\ell(m_2^\ell) = \mathrm{dst}_{t'_2}^\ell(m_2^\ell) = (2d_2$ or $2d_2 + 1)$.

Therefore, according to the lemma B.1, $\mathrm{dst}_{t_1}^{\ell+1}(m_1) = d_1 = d_2 = \mathrm{dst}_{t_2}^{\ell+1}(m_2)$.

Moreover, $\mathrm{Mid}_{t'_1}^\ell(m_2^\ell)$ and $\mathrm{Brd}_{t'_2}^\ell(b_2^\ell)$ so, according to the lemma 5.10, we have $\mathrm{LC}_{t'_1}^{\ell+1}(b_2, b'_2)$.

Therefore, we prove $\mathrm{Mid}_{t_1}^{\ell+1}(m_2)$ by case on the parity of $b'_2 - b_2$, by using the lemma 5.5 or 5.6.

□

To alleviate the proof of the synchronization at the following section, we include here the last lemmas about the middle field:

Lemma 6.4 (Three true middles cannot be adjacent)**.**

$$\forall \ell t c, \neg\, \mathrm{Brd}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c-1) \wedge \mathrm{Mid}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c+1) \Rightarrow \mathrm{False}$$

Proof. The proof is made p.46 by using the proposition 6.3. □

Lemma 6.5 (A true middle adjacent to a border has a distance = 1)**.**

$$\forall \ell t c, \neg\, \mathrm{Brd}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c) \wedge \Big(\mathrm{Brd}_t^\ell(c-1) \vee \mathrm{Brd}_t^\ell(c+1)\Big) \Rightarrow \mathrm{dst}_t^\ell(c) = 1$$

Proof. The proof is made p.47. □

Unfortunately, the last three lemmas remain to be proved:

Lemma 6.6 (Middles have max distance)**.**

$$\forall \ell t m, \neg\, \mathrm{Brd}_t^\ell(m) \wedge \mathrm{Mid}_t^\ell(m) \Rightarrow \Big(\forall c, \mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_t^\ell(m)\Big)$$

Lemma 6.7 (Cells with the same distance than a Middle are Middles)**.**

$$\forall \ell t m, \neg\, \mathrm{Brd}_t^\ell(m) \wedge \mathrm{Mid}_t^\ell(m) \Rightarrow \Big(\forall c, \mathrm{dst}_t^\ell(c) = \mathrm{dst}_t^\ell(m) \Rightarrow \mathrm{Mid}_t^\ell(c)\Big)$$

Lemma 6.8 (Middles appear when each cell is stable)**.**

$$\forall \ell t m, \neg\, \mathrm{Brd}_t^\ell(m) \wedge \mathrm{Mid}_t^\ell(m) \Rightarrow \forall c, \mathrm{Sta}_t^\ell(c)$$

7 Synchronization

For every layer, a region is split into two halves by the middle field, and they become two regions at the next layer. Therefore, from a layer to the next layer, the size of the regions is divided by 2, until every cell becomes a border.

We define the **output** field $\mathrm{Out}_{t+1}^\ell(c)$ as true for a cell c at a layer ℓ if its neighbors and itself are borders at the previous date t, and we say that a cell c fires at the date t if there exists one layer ℓ such that $\mathrm{Out}_t^\ell(c)$:

Definition 7.1 (Output Field)**.**

$$
\begin{aligned}
\mathrm{Out}_0^\ell(c) &\stackrel{\text{def}}{=} \text{False} \\
\mathrm{Out}_{t+1}^\ell(c) &\stackrel{\text{def}}{=} \mathrm{Brd}_t^\ell(c-1) \wedge \mathrm{Brd}_t^\ell(c) \wedge \mathrm{Brd}_t^\ell(c+1)
\end{aligned} \tag{8}
$$

The aim of the paper is to prove the theorem 7.4, which states that the output field is synchronized. In other words, that if a cell fires then every cell fires at the same time.

Lemma 7.2 (The Output Field fires for every layer)**.**

$$\forall \ell t c, \mathrm{Out}_{t+1}^\ell(c) \Rightarrow \mathrm{Out}_{t+1}^{\ell+1}(c)$$

Proof. Let ℓ, t and c.

The hypothesis $\mathrm{Out}_{t+1}^\ell(c)$ implies (8) that $\mathrm{Brd}_t^\ell(c-1)$ and $\mathrm{Brd}_t^\ell(c)$ and $\mathrm{Brd}_t^\ell(c+1)$.

So (2), we have $\mathrm{Brd}_t^{\ell+1}(c-1)$ and $\mathrm{Brd}_t^{\ell+1}(c)$ and $\mathrm{Brd}_t^{\ell+1}(c+1)$.

Therefore (8) we proved $\mathrm{Out}_{t+1}^{\ell+1}(c)$. □

Proposition 7.3 (Every cell will fire)**.**

$$\exists \ell t, \forall c, \mathrm{Out}_t^\ell(c)$$

Proof. According to the lemma 5.2 p.11, there exists t such that $\mathrm{LC}_t^0(1,n)$.

We remind that for a Light Cone $\mathrm{LC}_t^\ell(b_1,b_2)$, $b_2 - b_1 + 1$ is the number of cells forming the Light Cone, borders included.

According to the proposition 5.11 p.14, if $b_2 - b_1 + 1 \geq 5$, then the Light Cone is split in half at the next layer:

$$\mathrm{LC}_{t+\frac{b_2-b_1+1}{2}}^{\ell+1}\left(b_1, \frac{b_1+b_2}{2}\right) \wedge \mathrm{LC}_{t+\frac{b_2-b_1+1}{2}}^{\ell+1}\left(\frac{b_1+b_2+1}{2}, b_2\right)$$

Both new Light Cones are smaller : they have $\frac{b_2-b_1}{2}+1$ cells, borders included.

According to the axiom 3.1 p.4, we have $n \geq 3$, so $1 \leq \log_2(n-1)$, therefore there exists $\ell \in \mathbb{N}$ such that $\log_2(n-1) < \ell + 2$. Let ℓ be the smallest, so we have :

$$
\begin{aligned}
\log_2(n-1) < \ell + 2 &\Rightarrow n-1 < 2^\ell \times 4 \\
&\Rightarrow \tfrac{n-1}{2^\ell} + 1 < 5
\end{aligned}
$$

Therefore, we can repeat the process of splitting $\mathrm{LC}_t^0(1,n)$ ℓ times[1] until we obtain Light Cones with a number of cells ≤ 4. Moreover, according to the definition (7) of a Light Cone, the number of cells is ≥ 3. So, there is 3 or 4 cells in the last Light Cones.

According to the corollary 5.5 p.12 (for 3 cells) or the corolloray 5.6 p.13 (for 4 cells), in any case at this layer ℓ every cell becomes a border or a middle.

Therefore, at the layer $\ell + 1$, every cell becomes a border, then fires. □

Theorem 7.4 (The fire is synchronized)**.**

$$\forall \ell t c, \mathrm{Out}_t^\ell(c) \Rightarrow \forall c', \mathrm{Out}_t^\ell(c')$$

Proof. We prove $\forall t \ell c, \mathrm{Out}_t^\ell(c) \Rightarrow \forall c', \mathrm{Out}_t^\ell(c')$ by case on t:

[1] May be less, because $\frac{b_2-b_1}{2}$ is rounded down ?

- If $t = 0$, let ℓ and c. By (8), $\mathrm{Out}_0^\ell(c)$ is False, so the implication holds.
- Else, $t = t' + 1$ and we prove $\forall \ell c, \mathrm{Out}_{t'+1}^\ell(c) \Rightarrow \forall c', \mathrm{Out}_{t'+1}^\ell(c')$ by induction on ℓ:
  - $\mathrm{Out}_{t'+1}^0(c)$ implies (8) $\mathrm{Brd}_{t'}^0(c-1)$ and $\mathrm{Brd}_{t'}^0(c)$ and $\mathrm{Brd}_{t'}^0(c+1)$, so (2) we have $c - 1 = 1 \vee c - 1 = n$ and $c = 1 \vee c = n$ and $c + 1 = 1 \vee c + 1 = n$, which leads to a contradiction (three variables with distinct values, but only two available values).
  - We assume the induction hypothesis:

$$\forall c, \mathrm{Out}_{t'+1}^\ell(c) \Rightarrow \forall c', \mathrm{Out}_{t'+1}^\ell(c') \qquad (IH_\ell)$$

    Let c. The hypothesis $\mathrm{Out}_{t'+1}^{\ell+1}(c)$ implies (8) that $\mathrm{Brd}_{t'}^{\ell+1}(c-1)$ and $\mathrm{Brd}_{t'}^{\ell+1}(c)$ and $\mathrm{Brd}_{t'}^{\ell+1}(c+1)$.

    For each cell $c' \in \{c-1, c, c+1\}$, by using the lemma B.2 we have $\mathrm{Brd}_{t'}^\ell(c') \vee \neg \mathrm{Brd}_{t'}^\ell(c')$. But because $\mathrm{Brd}_{t'}^{\ell+1}(c')$ implies (2) that $\mathrm{Brd}_{t'}^\ell(c') \vee \mathrm{Mid}_{t'}^\ell(c')$, we have two cases: $\mathrm{Brd}_{t'}^\ell(c')$ or $\neg \mathrm{Brd}_{t'}^\ell(c') \wedge \mathrm{Mid}_{t'}^\ell(c')$.

    We prove $\forall c', \mathrm{Out}_{t'+1}^{\ell+1}(c')$ for the eight possible cases:

    * If $\mathrm{Brd}_{t'}^\ell(c-1)$ and $\mathrm{Brd}_{t'}^\ell(c)$ and $\mathrm{Brd}_{t'}^\ell(c+1)$ then (8) $\mathrm{Out}_{t'+1}^\ell(c)$.
      So, by using IH_ℓ we have for every c' that $\mathrm{Out}_{t'+1}^\ell(c')$.
      Therefore, by using the lemma 7.2, we have $\mathrm{Out}_{t'+1}^{\ell+1}(c')$.
    * If $\mathrm{Mid}_{t'}^\ell(c-1)$ and $\mathrm{Mid}_{t'}^\ell(c)$ and $\mathrm{Mid}_{t'}^\ell(c+1)$, we obtain a contradiction by using the lemma 6.4.
    * The other cases are :
      · $\mathrm{Brd}_{t'}^\ell(c-1)$ and $\mathrm{Brd}_{t'}^\ell(c)$ and $\mathrm{Mid}_{t'}^\ell(c+1)$
      · $\mathrm{Brd}_{t'}^\ell(c-1)$ and $\mathrm{Mid}_{t'}^\ell(c)$ and $\mathrm{Brd}_{t'}^\ell(c+1)$
      · $\mathrm{Brd}_{t'}^\ell(c-1)$ and $\mathrm{Mid}_{t'}^\ell(c)$ and $\mathrm{Mid}_{t'}^\ell(c+1)$
      · $\mathrm{Mid}_{t'}^\ell(c-1)$ and $\mathrm{Brd}_{t'}^\ell(c)$ and $\mathrm{Brd}_{t'}^\ell(c+1)$
      · $\mathrm{Mid}_{t'}^\ell(c-1)$ and $\mathrm{Brd}_{t'}^\ell(c)$ and $\mathrm{Mid}_{t'}^\ell(c+1)$
      · $\mathrm{Mid}_{t'}^\ell(c-1)$ and $\mathrm{Mid}_{t'}^\ell(c)$ and $\mathrm{Brd}_{t'}^\ell(c+1)$

      In every case, there exists a cell $m \in \{c-1, c, c+1\}$ with is a middle, not a border, and is adjacent to a border. So, by using the lemma 6.5 we have $\mathrm{dst}_{t'}^\ell(m) = 1$.
      Let c' be a cell. By using the lemma 6.6 we have that:

$$\mathrm{dst}_{t'}^\ell(c') \leq \mathrm{dst}_{t'}^\ell(m) = 1$$

      We prove that $\mathrm{Brd}_{t'}^{\ell+1}(c')$ by case on $\mathrm{dst}_{t'}^\ell(c')$:
      · In the case $\mathrm{dst}_{t'}^\ell(c') = 0$, by using the lemma 6.8 we have that $\mathrm{Sta}_{t'}^\ell(c')$, so by using the lemma B.9 we have that $\mathrm{Brd}_{t'}^\ell(c')$.
      · If $\mathrm{dst}_{t'}^\ell(c') = 1$, by using the lemma 6.7 we have that $\mathrm{Mid}_{t'}^\ell(c')$.

      Therefore, in every case $\mathrm{Brd}_{t'}^{\ell+1}(c')$.
      We proved it for every cell c', so we have $\mathrm{Brd}_{t'}^{\ell+1}(c'-1)$ and $\mathrm{Brd}_{t'}^{\ell+1}(c')$ and $\mathrm{Brd}_{t'}^{\ell+1}(c'+1)$, therefore $\mathrm{Out}_{t'+1}^{\ell+1}(c')$.

□

8 Conclusion

Therefore, by using only the axioms and definitions of the fields at the section 3 p.4, we proved the theorem 7.4 p.17 stating that as expected the fire is synchronized.

Unfortunaltely, the proof is not complete. It remains to prove the properties 6.6, 6.7 and 6.8 of the middles p.16. It also remains to implement most of the proofs in Coq, but some have been at the section F p.48. At least, this technical report attests the faisability of the task.

Moreover, we used simplified definitions in this paper, which does not take into account the cells outside the space. Indeed, in our framework, if $c = 1$ then the cell $c - 1$ is not in our space, neither $c + 1$ if $c = n$. We discuss this further in the appendix at the section A p.20.

Finally, this report proves the correctness of the high-level automaton by using a potentially infinite number of states, due to the potentially infinite number of layers and values for the distance field. Therefore, it remains also to formalize the construction of an explicit and finite table of states, as in [**?**].

A Neighborhood

We remind the formal definition of the fields (see the section 3 p.4) at the table 10, and discuss here what is required to be modified in order to solve the problem that if $c = 1$ then the cell $c - 1$ is not in our space, neither $c + 1$ for $c = n$.

The $c - 1$ and $c + 1$ in $\mathrm{Inp}_{t+1}(c)$ are a problem, so the field must be modified for the special case $c \in \{1, n\}$ with $\mathrm{Inp}_{t+1}(0) \stackrel{\text{def}}{=} \mathrm{Inp}_t(0) \vee \mathrm{Inp}_t(1)$ and $\mathrm{Inp}_{t+1}(n) \stackrel{\text{def}}{=} \mathrm{Inp}_t(n-1) \vee \mathrm{Inp}_t(n)$.

The $c - 1$ and $c + 1$ in $\mathrm{Ins}^{\ell+1}_{t+1}(c)$ are not a problem, because the algorithm verifies if $\mathrm{Ins}^{\ell}_{t+1}(c)$ first, which is false for $c \in \{1, n\}$.

The $c - 1$ and $c + 1$ in $\mathrm{dst}^{\ell}_{t+1}(c)$ are not a problem, because a cell $c \in \{1, n\}$ is not awaken or is a border, so in any case we have $\neg \mathrm{Ins}^{\ell}_{t+1}(c)$, so $\mathrm{dst}^{\ell}_{t+1}(c) = 0$.

The $c - 1$ and $c + 1$ in $\mathrm{Sta}^{\ell}_{t+1}(c)$ are not really a problem, because the algorithm verifies if $\mathrm{Brd}^{\ell}_{t+1}(c)$ first, which is true for $c \in \{1, n\}$ if the cell is awaken, and otherwise its distance is 0 so it cannot be $1 + \ldots$ But rigorously, it may be necessary to specify in the implementation the special case for $c \in \{1, n\}$ that $\mathrm{Sta}^{\ell}_{t+1}(c)$ only if $\mathrm{Brd}^{\ell}_{t+1}(c)$.

The $c - 1$ and $c + 1$ in $\mathrm{Mid}^{\ell}_{t+1}(c)$ are a problem, because borders become middles too in the last layer. But the confusion between "true" middles and middles which are also borders is not really intuitive and does not matter because $\mathrm{Brd}^{\ell+1}_{t}(c) \stackrel{\text{def}}{=} \mathrm{Brd}^{\ell}_{t}(c) \vee \mathrm{Mid}^{\ell}_{t}(c)$ and the output field is defined on the border field. So, instead of stating a special case, it may be less confusing to define $\mathrm{Mid}^{\ell}_{t+1}(c)$ as $\mathrm{Ins}^{\ell}_{t+1}(c) \wedge \ldots$ following by the rest of the definition, which solves in passing the problem for $c \in \{1, n\}$.

B Technical Lemmas

The proof of the following lemmas are very detailed and written in a Coq style, in order to be more easily implemented.

The following lemma is not really part of our framework, but has been written in order to be used in other lemmas:

Lemma B.1 (Equality up to Parity)**.**

$$\forall n d_1 d_2, (n = 2d_1 \vee n = 2d_1 + 1) \wedge (n = 2d_2 \vee n = 2d_2 + 1) \Rightarrow d_1 = d_2$$

Proof. The proof is made by case on n:

- In this case, n is even.

  Because $n = 2d_1 \vee n = 2d_1 + 1$ and n is even, we have that $n = 2d_1$.

  Because $n = 2d_2 \vee n = 2d_2 + 1$ and n is even, we have that $n = 2d_2$.

  So $2d_1 = 2d_2$, therefore $d_1 = d_2$.

- In this case, n is odd.

  Because $n = 2d_1 \vee n = 2d_1 + 1$ and n is odd, we have that $n = 2d_1 + 1$.

  Because $n = 2d_2 \vee n = 2d_2 + 1$ and n is odd, we have that $n = 2d_2 + 1$.

  So $2d_1 + 1 = 2d_2 + 1$, therefore $d_1 = d_2$.

□

The other lemmas are technical lemmas stating basic properties about the fields:

$$
\begin{aligned}
\mathrm{Inp}_0(c) &\stackrel{\text{def}}{=} \mathrm{gen}(c) = \mathrm{true} \\
\mathrm{Inp}_{t+1}(c) &\stackrel{\text{def}}{=} \mathrm{Inp}_t(c-1) \vee \mathrm{Inp}_t(c) \vee \mathrm{Inp}_t(c+1)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Brd}_t^0(c) &\stackrel{\text{def}}{=} \mathrm{Inp}_t(c) \wedge (1 = c \vee c = n) \\
\mathrm{Brd}_t^{\ell+1}(c) &\stackrel{\text{def}}{=} \mathrm{Brd}_t^\ell(c) \vee \mathrm{Mid}_t^\ell(c)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ins}_t^0(c) &\stackrel{\text{def}}{=} \mathrm{Inp}_t(c) \wedge 1 < c \wedge c < n \\
\mathrm{Ins}_0^{\ell+1}(c) &\stackrel{\text{def}}{=} \mathrm{False} \\
\mathrm{Ins}_{t+1}^{\ell+1}(c) &\stackrel{\text{def}}{=} \mathrm{Ins}_{t+1}^\ell(c) \wedge \mathrm{Sta}_{t+1}^\ell(c) \\
&\quad \wedge \Big(\mathrm{dst}_{t+1}^\ell(c) < \mathrm{dst}_t^\ell(c-1) \vee \mathrm{dst}_{t+1}^\ell(c) < \mathrm{dst}_t^\ell(c+1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{dst}_0^\ell(c) &\stackrel{\text{def}}{=} 0 \\
\mathrm{dst}_{t+1}^\ell(c) &\stackrel{\text{def}}{=} \begin{cases} 1 + \min\Big(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\Big) & \text{if } \mathrm{Ins}_{t+1}^\ell(c) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Sta}_0^\ell(c) &\stackrel{\text{def}}{=} \mathrm{Brd}_0^\ell(c) \\
\mathrm{Sta}_{t+1}^\ell(c) &\stackrel{\text{def}}{=} \mathrm{Brd}_{t+1}^\ell(c) \\
&\quad \vee \Big(\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c-1) \wedge \mathrm{Sta}_t^\ell(c-1)\Big) \\
&\quad \vee \Big(\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c+1) \wedge \mathrm{Sta}_t^\ell(c+1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Mid}_0^\ell(c) &\stackrel{\text{def}}{=} \mathrm{False} \\
\mathrm{Mid}_{t+1}^\ell(c) &\stackrel{\text{def}}{=} \Bigg(\mathrm{dst}_{t+1}^\ell(c) > \max\Big(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\Big) \\
&\qquad \wedge \mathrm{Sta}_t^\ell(c-1) \wedge \mathrm{Sta}_t^\ell(c+1)\Bigg) \\
&\quad \vee \Bigg(\mathrm{dst}_{t+1}^\ell(c) = \max\Big(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\Big) \\
&\qquad \wedge \mathrm{Sta}_t^\ell(c-1) \wedge \mathrm{Sta}_t^\ell(c) \wedge \mathrm{Sta}_t^\ell(c+1)\Bigg)
\end{aligned}
$$

Table 10: Formal Definition of the Fields

Lemma B.2 (The Border Field is True or False)**.**

$$\forall \ell t c, \mathrm{Brd}_t^\ell(c) \vee \neg\, \mathrm{Brd}_t^\ell(c)$$

Proof. By using the border field equations (2), or the characterization of bool/Prop fields (see the Coq file p.48). □

Lemma B.3 (Local Distance)**.**

$$\forall \ell t c, \mathrm{dst}^{\ell}_{t+1}(c) \leq 1 + \min\left(\mathrm{dst}^{\ell}_{t}(c-1), \mathrm{dst}^{\ell}_{t}(c+1)\right)$$

Proof. Let ℓ, t and c. By case :

- If $\mathrm{Ins}^{\ell}_{t+1}(c)$ then (6) the equality holds, so does the inequality.
- If $\neg\,\mathrm{Ins}^{\ell}_{t+1}(c)$ then (6) $\mathrm{dst}^{\ell}_{t+1}(c) = 0$, so the inequality holds.

□

Lemma B.4 (Middle Distance)**.**

$$\forall \ell t c, \mathrm{Mid}^{\ell}_{t+1}(c) \Rightarrow \mathrm{dst}^{\ell}_{t+1}(c) \geq \max\left(\mathrm{dst}^{\ell}_{t}(c-1), \mathrm{dst}^{\ell}_{t}(c+1)\right)$$

Proof. Let ℓ, t and c. By using (5), $\mathrm{Mid}^{\ell}_{t+1}(c)$ implies two cases:

$$\mathrm{dst}^{\ell}_{t+1}(c) > \max\left(\mathrm{dst}^{\ell}_{t}(c-1), \mathrm{dst}^{\ell}_{t}(c+1)\right)$$

$$\mathrm{dst}^{\ell}_{t+1}(c) = \max\left(\mathrm{dst}^{\ell}_{t}(c-1), \mathrm{dst}^{\ell}_{t}(c+1)\right)$$

and the result holds in every cases. □

Remark. We could use the previous lemma to simplify the proof of the following.

Lemma B.5 (Brd and Ins are exclusive)**.**

$$\forall \ell t c, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{False}$$

Proof. The proof is made by induction on ℓ:

- If $\ell = 0$ then $\mathrm{Brd}^{\ell}_{t}(c)$ implies (2) that $1 = c$ or $c = n$, and $\mathrm{Ins}^{\ell}_{t}(c)$ implies (3) that $1 < c < n$, hence the contradiction.
- We assume that:

  $$\forall t c, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{False} \qquad (IH_\ell)$$

  Let t and c, and we assume that:

  $$\mathrm{Brd}^{\ell+1}_{t}(c) \qquad (H_{\mathrm{Brd}})$$

  $$\mathrm{Ins}^{\ell+1}_{t}(c) \qquad (H_{\mathrm{Ins}})$$

  The proof of False is made by case on t :

  - If $t = 0$ then (3) $\mathrm{Ins}^{\ell+1}_{t}(c)$ is False, and is assumed.
  - If $t = t' + 1$, H_{Ins} implies (3) that:

    $$\mathrm{Ins}^{\ell}_{t'+1}(c) \qquad (H_{\mathrm{Ins}}2)$$

    $$\mathrm{dst}^{\ell}_{t'+1}(c) < \mathrm{dst}^{\ell}_{t'}(c-1) \vee \mathrm{dst}^{\ell}_{t'+1}(c) < \mathrm{dst}^{\ell}_{t'}(c-1) \qquad (H_{\mathrm{dst}})$$

    H_{Brd} implies (2) that $\mathrm{Brd}^{\ell}_{t'+1}(c) \vee \mathrm{Mid}^{\ell}_{t'+1}(c)$, so the proof is made by case:

* If $\mathrm{Brd}^{\ell}_{t'+1}(c)$, because $H_{\mathrm{Ins}}2$, we have False by using IH_{ℓ}.
* If $\mathrm{Mid}^{\ell}_{t'+1}(c)$, then by lemma B.4:

$$\mathrm{dst}^{\ell}_{t'+1}(c) \geq \max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big)$$

therefore $\mathrm{dst}^{\ell}_{t'+1}(c) \geq \mathrm{dst}^{\ell}_{t'}(c-1)$ and $\mathrm{dst}^{\ell}_{t'+1}(c) \geq \mathrm{dst}^{\ell}_{t'}(c+1)$, which contradicts H_{dst}.

□

Lemma B.6 (Distance of a Border)**.**

$$\forall \ell t c, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{dst}^{\ell}_{t}(c) = 0$$

Proof. Assuming that $\mathrm{Brd}^{\ell}_{t}(c)$, by using the lemma B.5, we have that $\neg\,\mathrm{Ins}^{\ell}_{t}(c)$. Therefore (6) $\mathrm{dst}^{\ell}_{t}(c) = 0$. □

Lemma B.7 (Middles have stable neighbours)**.**

$$\forall \ell t c, \mathrm{Mid}^{\ell}_{t+1}(c) \Rightarrow \mathrm{Sta}^{\ell}_{t}(c-1) \wedge \mathrm{Sta}^{\ell}_{t}(c+1)$$

Proof. The result is obtained by hypothesis on the two cases (5) of $\mathrm{Mid}^{\ell}_{t+1}(c)$. □

Remark. We could use the previous lemma (introduced lately during the redaction) to simplify some proofs.

Lemma B.8 (A middle is stable)**.**

$$\forall \ell t c, \mathrm{Mid}^{\ell}_{t}(c) \Rightarrow \mathrm{Sta}^{\ell}_{t}(c)$$

Proof. Let ℓ. The proof is made by case on t:

- If $t = 0$, let c. By (5), $\mathrm{Mid}^{\ell}_{0}(c)$ is False, so the implication holds.
- Else, we prove $\mathrm{Sta}^{\ell}_{t}(c)$ by case (5) on the hypothesis $\mathrm{Mid}^{\ell}_{t}(c)$:
  - In the first case we assume:

$$\mathrm{dst}^{\ell}_{t+1}(c) > \max\Big(\mathrm{dst}^{\ell}_{t}(c-1), \mathrm{dst}^{\ell}_{t}(c+1)\Big) \tag{Hd}$$

$$\mathrm{Sta}^{\ell}_{t}(c-1) \tag{HSL}$$

$$\mathrm{Sta}^{\ell}_{t}(c+1) \tag{HSR}$$

Hd implies that:

$$\begin{aligned}\mathrm{dst}^{\ell}_{t+1}(c) &\geq 1 + \max\Big(\mathrm{dst}^{\ell}_{t}(c-1), \mathrm{dst}^{\ell}_{t}(c+1)\Big)\\ &\geq 1 + \mathrm{dst}^{\ell}_{t}(c-1)\end{aligned}$$

And the lemma B.3 implies that:

$$\begin{aligned}\mathrm{dst}^{\ell}_{t+1}(c) &\leq 1 + \min\Big(\mathrm{dst}^{\ell}_{t}(c-1), \mathrm{dst}^{\ell}_{t}(c+1)\Big)\\ &\leq 1 + \mathrm{dst}^{\ell}_{t}(c-1)\end{aligned}$$

So $\mathrm{dst}^{\ell}_{t+1}(c) = 1 + \mathrm{dst}^{\ell}_{t}(c-1)$. But *HSL*, therefore (4) $\mathrm{Sta}^{\ell}_{t}(c)$.

– In the second case, $\mathrm{Sta}_t^\ell(c)$ is obtained by hypothesis.

□

Lemma B.9 (A stable cell with dst = 0 is a border)**.**

$$\forall \ell t c, \mathrm{Sta}_t^\ell(c) \wedge \mathrm{dst}_t^\ell(c) = 0 \Rightarrow \mathrm{Brd}_t^\ell(c)$$

Proof. Let ℓ. The proof is made by case on t:

- If $t = 0$, let c. We assume that $\mathrm{Sta}_0^\ell(c)$ and $\mathrm{dst}_0^\ell(c) = 0$. $\mathrm{Brd}_0^\ell(c)$ is obtained (4) with the hypothesis $\mathrm{Sta}_0^\ell(c)$.
- Else, let c. We assume that $\mathrm{Sta}_{t+1}^\ell(c)$ and $\mathrm{dst}_{t+1}^\ell(c) = 0$. The proof is made by case (4) on the hypothesis $\mathrm{Sta}_{t+1}^\ell(c)$:
  - In the first case $\mathrm{Brd}_{t+1}^\ell(c)$ is obtained by hypothesis.
  - In the second case we have $\mathrm{dst}_{t+1}^\ell(c) = 1+\mathrm{dst}_t^\ell(c-1)$, which contradicts $\mathrm{dst}_{t+1}^\ell(c) = 0$.
  - In the second case we have $\mathrm{dst}_{t+1}^\ell(c) = 1+\mathrm{dst}_t^\ell(c+1)$, which contradicts $\mathrm{dst}_{t+1}^\ell(c) = 0$.

□

Corollary B.10 (A non-border Middle has a distance > 0)**.**

$$\forall \ell t c, \neg \mathrm{Brd}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) > 0$$

Proof. By using the contraposition of the lemma B.9 on the hypothesis $\neg \mathrm{Brd}_t^\ell(c)$ we have $\neg \mathrm{Sta}_t^\ell(c)$ or $\mathrm{dst}_t^\ell(c) \neq 0$.

But by using the lemma B.8 on the hypothesis $\mathrm{Mid}_t^\ell(c)$ we have $\mathrm{Sta}_t^\ell(c)$.

So $\mathrm{dst}_t^\ell(c) > 0$. □

Lemma B.11 (At layer 0, the cells end up being awaken)**.**

$$\exists t, \forall c, \mathrm{Inp}_t(c)$$

Proof. By axiom 3.2, there exists at least one general, so by (1) there exists some cells awaken at $t = 0$. Then, the input field propagates from cell to cell, so the last cells are the most distant from the generals.

If a cell is between two generals g_1 and g_2, then it requires $\left\lceil \frac{|g_1-g_2|}{2} \right\rceil$ steps to be awakened. If a cell is between a general g and a border (included), then it requires $g-1$ steps on the left, and $n-g$ steps on the right.

Let $\{g_1, \ldots, g_k\} = \{1 \le c \le n \mid \mathrm{gen}(c) = \mathrm{true}\}$, with $g_1 \le \cdots \le g_k$. Therefore, we should prove in the Coq code that:

$$t = \max_{1 \le i < k} \left\{ g_1 - 1, \left\lceil \frac{g_{i+1} - g_i}{2} \right\rceil, n - g_k \right\}$$

□

C Monotonicity

In this section we prove monotonicity properties for the fields, which means that if the property holds for a given t, then it holds for every $t' \geq t$.

Lemma C.1 (Inp is monotone)**.**

$$\forall tc, \mathrm{Inp}_t(c) \Rightarrow (\forall t', t' \geq t \Rightarrow \mathrm{Inp}_{t'}(c))$$

Proof. Let t and c. We assume the hypothesis $\mathrm{Inp}_t(c)$.

Let t'. We prove $\mathrm{Inp}_{t'}(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{Inp}_{t'}(c)$ by hypothesis.
- If $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{Inp}_{t''}(c)$, then by using the equation (1) we have $\mathrm{Inp}_{t''+1}(c)$.

Therefore $\mathrm{Inp}_{t'}(c)$. □

Lemma C.2 (Ins monotone implies dst is increasing)**.**

$$\forall \ell, \left(\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\right) \Rightarrow \left(\forall tc, \mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_{t+1}^\ell(c)\right)$$

Proof. Let ℓ, and we assume:

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \tag{H_{ins}}$$

The proof is made by induction on t:

- If $t = 0$, then (6) $\mathrm{dst}_t^\ell(c) = 0$, therefore $\mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_{t+1}^\ell(c)$.
- We assume that:

  $$\forall c, \mathrm{dst}_t^\ell(c) \leq \mathrm{dst}_{t+1}^\ell(c) \tag{IH_t}$$

  Let c. We proove by case that $\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$:

  – If $\mathrm{Ins}_{t+2}^\ell(c)$ then (6) $\mathrm{dst}_{t+2}^\ell(c) = 1 + \min\left(\mathrm{dst}_{t+1}^\ell(c-1), \mathrm{dst}_{t+1}^\ell(c+1)\right)$.

  But by using IH_t we have that $\mathrm{dst}_t^\ell(c-1) \leq \mathrm{dst}_{t+1}^\ell(c-1)$ and $\mathrm{dst}_t^\ell(c+1) \leq \mathrm{dst}_{t+1}^\ell(c+1)$, so:

  $$1 + \min\left(\mathrm{dst}_t^\ell(c-1), \mathrm{dst}_t^\ell(c+1)\right) \leq \mathrm{dst}_{t+2}^\ell(c)$$

  Therefore, by using the lemma B.3, we have $\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$.

  – If $\neg\,\mathrm{Ins}_{t+2}^\ell(c)$ then (6) $\mathrm{dst}_{t+2}^\ell(c) = 0$. Moreover, by using the contraposition of H_{ins} we have $\neg\,\mathrm{Ins}_{t+1}^\ell(c)$, so $\mathrm{dst}_{t+1}^\ell(c) = 0$ too. Therefore, in any cases, $\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$.

□

Lemma C.3 (Brd and Ins monotone implies a stable dst is constant)**.**

$$\forall \ell, \left(\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c)\right) \Rightarrow \left(\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\right)$$

$$\Rightarrow \left(\forall tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) = \mathrm{dst}_{t+1}^\ell(c)\right)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c) \tag{H_{Brd}}$$

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \tag{H_{Ins}}$$

We prove $\forall tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) = \mathrm{dst}_{t+1}^\ell(c)$ by induction on t:

- If $t = 0$ then (4) the hypothesis $\mathrm{Sta}_0^\ell(c)$ implies $\mathrm{Brd}_0^\ell(c)$, so according to H_{Brd} we have $\mathrm{Brd}_1^\ell(c)$ too. Therefore, according to the lemma B.6, we have $\mathrm{dst}_0^\ell(c) = 0 = \mathrm{dst}_1^\ell(c)$.
- We assume the induction hypothesis:

  $$\forall c, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{dst}_t^\ell(c) = \mathrm{dst}_{t+1}^\ell(c) \tag{IH_t}$$

  Let c. We assume the hypothesis:

  $$\mathrm{Sta}_{t+1}^\ell(c) \tag{H_{Sta}}$$

  We prove $\mathrm{dst}_{t+1}^\ell(c) = \mathrm{dst}_{t+2}^\ell(c)$ by case (4) on H_{Sta}:

  – If $\mathrm{Brd}_{t+1}^\ell(c)$ then according to H_{Brd} we have $\mathrm{Brd}_{t+2}^\ell(c)$ too. Therefore, according to the lemma B.6, we have $\mathrm{dst}_{t+1}^\ell(c) = 0 = \mathrm{dst}_{t+2}^\ell(c)$.

  – In that case, we have:

  $$\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c-1) \tag{H_{dst}}$$

  $$\mathrm{Sta}_t^\ell(c-1) \tag{$H_{\mathrm{Sta}}2$}$$

  Firstly, by using $H_{\mathrm{Sta}}2$ and the induction hypothesis IH_t we have $\mathrm{dst}_t^\ell(c-1) = \mathrm{dst}_{t+1}^\ell(c-1)$, so by using H_{dst}, we have :

  $$\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c-1) = 1 + \mathrm{dst}_{t+1}^\ell(c-1)$$

  Moreover, by using the lemma B.3, we have:

  $$\begin{aligned}\mathrm{dst}_{t+2}^\ell(c) &\leq 1 + \min\left(\mathrm{dst}_{t+1}^\ell(c-1), \mathrm{dst}_{t+1}^\ell(c+1)\right)\\ &\leq 1 + \mathrm{dst}_{t+1}^\ell(c-1)\\ &\leq \mathrm{dst}_{t+1}^\ell(c)\end{aligned}$$

  Secondly, by using H_{Ins} and the lemma C.2:

  $$\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$$

  Therefore, we proved the equality.

  – If $\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c+1)$ and $\mathrm{Sta}_t^\ell(c+1)$, the proof is similar to the previous case.

$\square$

Lemma C.4 (Brd and Ins monotone implies Sta monotone)**.**

$$\forall \ell, \left(\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c)\right) \Rightarrow \left(\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\right)$$

$$\Rightarrow \left(\forall tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{Sta}_{t+1}^\ell(c)\right)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c) \tag{H_{Brd}}$$

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \tag{H_{Ins}}$$

We prove $\forall tc, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{Sta}_{t+1}^\ell(c)$ by induction on t:

- If $t = 0$ then (4) the hypothesis $\mathrm{Sta}_0^\ell(c)$ implies $\mathrm{Brd}_0^\ell(c)$, so according to H_{Brd} we have $\mathrm{Brd}_1^\ell(c)$ too. Therefore, we have (4) the first case of $\mathrm{Sta}_1^\ell(c)$.
- We assume the induction hypothesis:

$$\forall c, \mathrm{Sta}_t^\ell(c) \Rightarrow \mathrm{Sta}_{t+1}^\ell(c) \tag{IH_t}$$

  Let c. We assume the hypothesis:

$$\mathrm{Sta}_{t+1}^\ell(c) \tag{H_{Sta}}$$

  We prove $\mathrm{Sta}_{t+2}^\ell(c)$ by case (4) on H_{Sta}:

  – If $\mathrm{Brd}_{t+1}^\ell(c)$ then according to H_{Brd} we have $\mathrm{Brd}_{t+2}^\ell(c)$ too. Therefore, we have (4) the first case of $\mathrm{Sta}_{t+2}^\ell(c)$.

  – In that case, we have:

$$\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c-1) \tag{H_{dst}}$$

$$\mathrm{Sta}_t^\ell(c-1) \tag{$H_{\mathrm{Sta}}2$}$$

  By using H_{Brd}, H_{Ins} and the lemma C.3, $H_{\mathrm{Sta}}2$ implies that:

$$\mathrm{dst}_t^\ell(c-1) = \mathrm{dst}_{t+1}^\ell(c-1) \tag{H}$$

  Firstly, by using the lemma B.3 then H then H_{dst}, we have:

$$\begin{aligned}\mathrm{dst}_{t+2}^\ell(c) &\leq 1 + \min\Big(\mathrm{dst}_{t+1}^\ell(c-1), \mathrm{dst}_{t+1}^\ell(c+1)\Big)\\ &\leq 1 + \mathrm{dst}_{t+1}^\ell(c-1)\\ &= 1 + \mathrm{dst}_t^\ell(c-1)\\ &= \mathrm{dst}_{t+1}^\ell(c)\end{aligned}$$

  Secondly, by using H_{Ins} and the lemma C.2, we have:

$$\mathrm{dst}_{t+1}^\ell(c) \leq \mathrm{dst}_{t+2}^\ell(c)$$

  Therefore $\mathrm{dst}_{t+1}^\ell(c) = \mathrm{dst}_{t+2}^\ell(c)$. So, by using H_{dst} then H:

$$\begin{aligned}\mathrm{dst}_{t+2}^\ell(c) &= \mathrm{dst}_{t+1}^\ell(c)\\ &= 1 + \mathrm{dst}_t^\ell(c-1)\\ &= 1 + \mathrm{dst}_{t+1}^\ell(c-1)\end{aligned}$$

  Moreover, by using $H_{\mathrm{Sta}}2$ and the induction hypothesis IH_t we have $\mathrm{Sta}_{t+1}^\ell(c-1)$. Therefore (4) we proved $\mathrm{Sta}_{t+2}^\ell(c)$.

  – If $\mathrm{dst}_{t+1}^\ell(c) = 1 + \mathrm{dst}_t^\ell(c+1)$ and $\mathrm{Sta}_t^\ell(c+1)$, the proof is similar to the previous case.

□

Lemma C.5 (Brd and Ins monotone implies Mid monotone)**.**

$$\forall \ell, \Big(\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c)\Big) \Rightarrow \Big(\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\Big)$$
$$\Rightarrow \Big(\forall tc, \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{Mid}_{t+1}^\ell(c)\Big)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c) \qquad (H_{\mathrm{Brd}})$$

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \qquad (H_{\mathrm{Ins}})$$

We prove $\forall tc, \mathrm{Mid}_t^\ell(c) \Rightarrow \mathrm{Mid}_{t+1}^\ell(c)$ by case on t:

- If $t = 0$ then (5) $\mathrm{Mid}_t^\ell(c)$ is False, so the implication holds.
- If $t = t' + 1$, let c, and we assume the hypothesis:

$$\mathrm{Mid}_{t'+1}^\ell(c) \qquad (H_{\mathrm{Mid}})$$

  We prove $\mathrm{Mid}_{t'+2}^\ell(c)$ by case (5) on H_{Mid}:

  – In the first case, we have:

$$\mathrm{dst}_{t'+1}^\ell(c) > \max\Big(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\Big) \qquad (H_{\mathrm{dst}})$$

$$\mathrm{Sta}_{t'}^\ell(c-1) \qquad (H_{\mathrm{Sta}}L)$$

$$\mathrm{Sta}_{t'}^\ell(c+1) \qquad (H_{\mathrm{Sta}}R)$$

  By using H_{Brd}, H_{Ins} and the lemma C.3:

  ∗ $H_{\mathrm{Sta}}L$ implies that $\mathrm{dst}_{t'}^\ell(c-1) = \mathrm{dst}_{t'+1}^\ell(c-1)$

  ∗ $H_{\mathrm{Sta}}R$ implies that $\mathrm{dst}_{t'}^\ell(c+1) = \mathrm{dst}_{t'+1}^\ell(c+1)$

  Therefore, we have:

$$\max\Big(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\Big) = \max\Big(\mathrm{dst}_{t'+1}^\ell(c-1), \mathrm{dst}_{t'+1}^\ell(c+1)\Big) \qquad (H_{\max})$$

  So, by using H_{Ins} and the lemma C.2, then H_{dst}, then $H_{\max}$, we have:

$$\begin{aligned}\mathrm{dst}_{t'+2}^\ell(c) &\geq \mathrm{dst}_{t'+1}^\ell(c)\\ &> \max\Big(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\Big)\\ &= \max\Big(\mathrm{dst}_{t'+1}^\ell(c-1), \mathrm{dst}_{t'+1}^\ell(c+1)\Big)\end{aligned}$$

  Moreover, by using H_{Brd}, H_{Ins} and the lemma C.4:

  ∗ $H_{\mathrm{Sta}}L$ implies that $\mathrm{Sta}_{t'+1}^\ell(c-1)$

  ∗ $H_{\mathrm{Sta}}R$ implies that $\mathrm{Sta}_{t'+1}^\ell(c+1)$

  Therefore, we have the left part of $\mathrm{Mid}_{t'+2}^\ell(c)$.

– In the second case, we have:

$$\mathrm{dst}^{\ell}_{t'+1}(c) = \max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big) \tag{H_{dst}}$$

$$\mathrm{Sta}^{\ell}_{t'}(c-1) \tag{$H_{\mathrm{Sta}}L$}$$

$$\mathrm{Sta}^{\ell}_{t'}(c) \tag{$H_{\mathrm{Sta}}C$}$$

$$\mathrm{Sta}^{\ell}_{t'}(c+1) \tag{$H_{\mathrm{Sta}}R$}$$

By using H_{Brd}, H_{Ins} and the lemma C.4:

  * $H_{\mathrm{Sta}}L$ implies that $\mathrm{Sta}^{\ell}_{t'+1}(c-1)$
  * $H_{\mathrm{Sta}}C$ implies that $\mathrm{Sta}^{\ell}_{t'+1}(c)$
  * $H_{\mathrm{Sta}}R$ implies that $\mathrm{Sta}^{\ell}_{t'+1}(c+1)$

Therefore, to obtain the right part of $\mathrm{Mid}^{\ell}_{t'+2}(c)$, it remains only to prove that $\mathrm{dst}^{\ell}_{t'+2}(c) = \max\Big(\mathrm{dst}^{\ell}_{t'+1}(c-1), \mathrm{dst}^{\ell}_{t'+1}(c+1)\Big)$.

By using H_{Brd}, H_{Ins} and the lemma C.3, $\mathrm{Sta}^{\ell}_{t'+1}(c)$ implies that:

$$\mathrm{dst}^{\ell}_{t'+1}(c) = \mathrm{dst}^{\ell}_{t'+2}(c) \tag{$H_{\mathrm{dst}}2$}$$

By using H_{Brd}, H_{Ins} and the lemma C.3:

  * $H_{\mathrm{Sta}}L$ implies that $\mathrm{dst}^{\ell}_{t'}(c-1) = \mathrm{dst}^{\ell}_{t'+1}(c-1)$
  * $H_{\mathrm{Sta}}R$ implies that $\mathrm{dst}^{\ell}_{t'}(c+1) = \mathrm{dst}^{\ell}_{t'+1}(c+1)$

Therefore, we have:

$$\max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big) = \max\Big(\mathrm{dst}^{\ell}_{t'+1}(c-1), \mathrm{dst}^{\ell}_{t'+1}(c+1)\Big) \tag{$H_{\max}$}$$

So, by using $H_{\mathrm{dst}}2$, then H_{dst}, then $H_{\max}$, we have:

$$\begin{aligned}\mathrm{dst}^{\ell}_{t'+2}(c) &= \mathrm{dst}^{\ell}_{t'+1}(c)\\ &= \max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big)\\ &= \max\Big(\mathrm{dst}^{\ell}_{t'+1}(c-1), \mathrm{dst}^{\ell}_{t'+1}(c+1)\Big)\end{aligned}$$

Therefore, we have the right part of $\mathrm{Mid}^{\ell}_{t'+2}(c)$.

□

Lemma C.6 (Brd^{ℓ} and Ins^{ℓ} monotone implies $\mathrm{Brd}^{\ell+1}$ monotone)**.**

$$\forall \ell, \Big(\forall tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell}_{t+1}(c)\Big) \Rightarrow \Big(\forall tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t+1}(c)\Big)$$

$$\Rightarrow \Big(\forall tc, \mathrm{Brd}^{\ell+1}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell+1}_{t+1}(c)\Big)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell}_{t+1}(c) \tag{H_{Brd}}$$

$$\forall tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t+1}(c) \tag{H_{Ins}}$$

Let t and c. We prove $\mathrm{Brd}^{\ell+1}_{t+1}(c)$ by case (2) on the hypothesis $\mathrm{Brd}^{\ell+1}_{t}(c)$:

- In the first case, we have $\mathrm{Brd}_t^\ell(c)$, so by using H_{Brd} we have $\mathrm{Brd}_{t+1}^\ell(c)$.

  Therefore (2), we proved the left part of $\mathrm{Brd}_{t+1}^{\ell+1}(c)$.

- In the second case, we have $\mathrm{Mid}_t^\ell(c)$.

  So, by using H_{Brd}, H_{Ins} and the lemma C.5 we have $\mathrm{Mid}_{t+1}^\ell(c)$.

  Therefore (2), we proved the right part of $\mathrm{Brd}_{t+1}^{\ell+1}(c)$.

□

Lemma C.7 (Brd^ℓ and Ins^ℓ monotone implies $\mathrm{Ins}^{\ell+1}$ monotone)**.**

$$\forall \ell, \Big(\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c)\Big) \Rightarrow \Big(\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c)\Big)$$

$$\Rightarrow \Big(\forall tc, \mathrm{Ins}_t^{\ell+1}(c) \Rightarrow \mathrm{Ins}_{t+1}^{\ell+1}(c)\Big)$$

Proof. Let ℓ. We assume that:

$$\forall tc, \mathrm{Brd}_t^\ell(c) \Rightarrow \mathrm{Brd}_{t+1}^\ell(c) \qquad (H_{\mathrm{Brd}})$$

$$\forall tc, \mathrm{Ins}_t^\ell(c) \Rightarrow \mathrm{Ins}_{t+1}^\ell(c) \qquad (H_{\mathrm{Ins}})$$

We prove $\mathrm{Ins}_t^{\ell+1}(c) \Rightarrow \mathrm{Ins}_{t+1}^{\ell+1}(c)$ by case on t:

- If $t = 0$ then (3) $\mathrm{Ins}_t^{\ell+1}(c)$ is False, so the implication holds.
- If $t = t' + 1$, let c. The hypothesis $\mathrm{Ins}_{t'+1}^{\ell+1}(c)$ implies (3):

$$\mathrm{Ins}_{t'+1}^\ell(c) \qquad (H_{\mathrm{Ins}}2)$$

$$\mathrm{Sta}_{t'+1}^\ell(c) \qquad (H_{\mathrm{Sta}})$$

$$\mathrm{dst}_{t'+1}^\ell(c) < \mathrm{dst}_{t'}^\ell(c-1) \vee \mathrm{dst}_{t'+1}^\ell(c) < \mathrm{dst}_{t'}^\ell(c-1) \qquad (H_{\mathrm{dst}})$$

  By using H_{Ins}, $H_{\mathrm{Ins}}2$ implies that $\mathrm{Ins}_{t'+2}^\ell(c)$.

  Moreover, by using H_{Brd}, H_{Ins} and the lemma C.4, H_{Sta} implies that $\mathrm{Sta}_{t'+2}^\ell(c)$.

  Therefore, to obtain $\mathrm{Ins}_{t'+2}^{\ell+1}(c)$, it remains only to prove that $\mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1) \vee \mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1)$.

  Notice that by using H_{Brd}, H_{Ins} and the lemma C.3, H_{Sta} implies that:

$$\mathrm{dst}_{t'+1}^\ell(c) = \mathrm{dst}_{t'+2}^\ell(c) \qquad (H)$$

  We prove $\mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1) \vee \mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1)$ by case on H_{dst}:

  - In the first case, we have $\mathrm{dst}_{t'+1}^\ell(c) < \mathrm{dst}_{t'}^\ell(c-1)$.
    So, by using H, then the case hypothesis, then H_{Ins} and the lemma C.2, we have:

$$\begin{aligned} \mathrm{dst}_{t'+2}^\ell(c) &= \mathrm{dst}_{t'+1}^\ell(c) \\ &< \mathrm{dst}_{t'}^\ell(c-1) \\ &\leq \mathrm{dst}_{t'+1}^\ell(c-1) \end{aligned}$$

    Therefore, we proved the left part of $\mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1) \vee \mathrm{dst}_{t'+2}^\ell(c) < \mathrm{dst}_{t'+1}^\ell(c-1)$.

– The case $\mathrm{dst}^{\ell}_{t'+1}(c) < \mathrm{dst}^{\ell}_{t'}(c+1)$ is similar, and proves the right part of $\mathrm{dst}^{\ell}_{t'+2}(c) < \mathrm{dst}^{\ell}_{t'+1}(c-1) \vee \mathrm{dst}^{\ell}_{t'+2}(c) < \mathrm{dst}^{\ell}_{t'+1}(c-1)$.

□

Proposition C.8 (Brd and Ins are monotone)**.**

$$\forall \ell, \Big(\forall tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell}_{t+1}(c)\Big) \wedge \Big(\forall tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t+1}(c)\Big)$$

Proof. The proof is made by induction on ℓ:

- If $\ell = 0$, we prove the two parts separately:
  - Let t and c. The hypothesis $\mathrm{Brd}^{0}_{t}(c)$ implies (2) that $\mathrm{Inp}_{t}(c)$ and $1 = c \vee c = n$.
    So, by using the lemma C.1, we have $\mathrm{Inp}_{t+1}(c)$ and $1 = c \vee c = n$.
    Therefore (2) we proved that $\mathrm{Brd}^{0}_{t+1}(c)$.
  - Let t and c. The hypothesis $\mathrm{Ins}^{0}_{t}(c)$ implies (3) that $\mathrm{Inp}_{t}(c)$ and $1 < c < n$.
    So, by using the lemma C.1, we have $\mathrm{Inp}_{t+1}(c)$ and $1 < c < n$.
    Therefore (3) we proved that $\mathrm{Ins}^{0}_{t+1}(c)$.
- We assume the induction hypothesis:

  $$\forall tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell}_{t+1}(c) \qquad (IH^{\ell}_{\mathrm{Brd}})$$

  $$\forall tc, \mathrm{Ins}^{\ell}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell}_{t+1}(c) \qquad (IH^{\ell}_{\mathrm{Ins}})$$

  By using IH^{ℓ}_{Brd}, IH^{ℓ}_{Ins} and the lemma C.6, we have:

  $$\forall tc, \mathrm{Brd}^{\ell+1}_{t}(c) \Rightarrow \mathrm{Brd}^{\ell+1}_{t+1}(c)$$

  By using IH^{ℓ}_{Brd}, IH^{ℓ}_{Ins} and the lemma C.7, we have:

  $$\forall tc, \mathrm{Ins}^{\ell+1}_{t}(c) \Rightarrow \mathrm{Ins}^{\ell+1}_{t+1}(c)$$

  Therefore, we proved the induction step.

□

Corollary C.9 (Brd is monotone)**.**

$$\forall \ell tc, \mathrm{Brd}^{\ell}_{t}(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Brd}^{\ell}_{t'}(c)\Big)$$

Proof. Let ℓ, t and c. We assume the hypothesis $\mathrm{Brd}^{\ell}_{t}(c)$.
Let t'. We prove $\mathrm{Brd}^{\ell}_{t'}(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{Brd}^{\ell}_{t'}(c)$ by hypothesis.
- If $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{Brd}^{\ell}_{t''}(c)$, then by using the left part of the proposition C.8 we have $\mathrm{Brd}^{\ell}_{t''+1}(c)$. Therefore $\mathrm{Brd}^{\ell}_{t'}(c)$.

□

Corollary C.10 (Ins is monotone)**.**

$$\forall \ell t c, \mathrm{Ins}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Ins}_{t'}^\ell(c)\Big)$$

Proof. The proof is similar to the previous one, and uses the right part of the proposition C.8. □

Corollary C.11 (Sta is monotone)**.**

$$\forall \ell t c, \mathrm{Sta}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Sta}_{t'}^\ell(c)\Big)$$

Proof. Let ℓ, t and c. We assume the hypothesis $\mathrm{Sta}_t^\ell(c)$.

Let t'. We prove $\mathrm{Sta}_{t'}^\ell(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{Sta}_{t'}^\ell(c)$ by hypothesis.
- If $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{Sta}_{t''}^\ell(c)$, then by using both parts of the proposition C.8 and the lemma C.4 we have $\mathrm{Sta}_{t''+1}^\ell(c)$.

  Therefore $\mathrm{Sta}_{t'}^\ell(c)$.

□

Corollary C.12 (Mid is monotone)**.**

$$\forall \ell t c, \mathrm{Mid}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{Mid}_{t'}^\ell(c)\Big)$$

Proof. The proof is similar to the previous one, and uses both parts of the proposition C.8 and the lemma C.5. □

Corollary C.13 (dst is increasing)**.**

$$\forall \ell t c t', t' \geq t \Rightarrow \mathrm{dst}_{t'}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$$

Proof. Let ℓ, t, c and t'.

We prove $\mathrm{dst}_{t'}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{dst}_{t'}^\ell(c) = \mathrm{dst}_t^\ell(c)$, therefore $\mathrm{dst}_{t'}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$.
- In that case $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{dst}_{t''}^\ell(c) \geq \mathrm{dst}_t^\ell(c)$.

  Therefore, by using the right part of the proposition C.8 and the lemma C.2, then the hypothesis, we have:

$$\begin{aligned} \mathrm{dst}_{t'}^\ell(c) &= \mathrm{dst}_{t''+1}^\ell(c) \\ &\geq \mathrm{dst}_{t''}^\ell(c) \\ &\geq \mathrm{dst}_t^\ell(c) \end{aligned}$$

□

Corollary C.14 (A stable dst is constant)**.**

$$\forall \ell t c, \mathrm{Sta}_t^\ell(c) \Rightarrow \Big(\forall t', t' \geq t \Rightarrow \mathrm{dst}_{t'}^\ell(c) = \mathrm{dst}_t^\ell(c)\Big)$$

Proof. Let ℓ, t and c. We assume the hypothesis $\mathrm{Sta}_t^\ell(c)$.

Let t'. We prove $\mathrm{Brd}_t^\ell(c)$ by case on the hypothesis $t' \geq t$:

- If $t' = t$ then $\mathrm{dst}^{\ell}_{t'}(c) = \mathrm{dst}^{\ell}_{t}(c)$.
- In that case $t' = t'' + 1$ with $t'' \geq t$ such that $\mathrm{dst}^{\ell}_{t''}(c) = \mathrm{dst}^{\ell}_{t}(c)$.

  By using the hypotheses $\mathrm{Sta}^{\ell}_{t}(c)$ and $t'' \geq t$, and the lemma C.11, we have $\mathrm{Sta}^{\ell}_{t''}(c)$.

  So, by using both parts of the proposition C.8 and the lemma C.3 we have $\mathrm{dst}^{\ell}_{t''}(c) = \mathrm{dst}^{\ell}_{t''+1}(c)$. Therefore:

$$\begin{aligned}\mathrm{dst}^{\ell}_{t'}(c) &= \mathrm{dst}^{\ell}_{t''+1}(c)\\ &= \mathrm{dst}^{\ell}_{t''}(c)\\ &= \mathrm{dst}^{\ell}_{t}(c)\end{aligned}$$

□

D Proofs for the Light Cones

The proofs of the following results are very detailed and written in a Coq style, in order to be implemented. Therefore, for sake of clarity, we will only write and comment the results in the section 5 p.11, and detail the proofs in this section.

Proposition (5.3 p.12: Running of a Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^{\ell}_{t}(b_1, b_2) \Rightarrow \forall\ 0 \leq d \leq \frac{b_2 - b_1}{2},$$

$$\mathrm{dst}^{\ell}_{t+d}(b_1 + d) = d \wedge \mathrm{Sta}^{\ell}_{t+d}(b_1 + d)$$

$$\wedge\, \mathrm{dst}^{\ell}_{t+d}(b_2 - d) = d \wedge \mathrm{Sta}^{\ell}_{t+d}(b_2 - d)$$

$$\wedge \Big(\forall\ b_1 + d \leq c \leq b_2 - d, \mathrm{dst}^{\ell}_{t+d}(c) \geq d\Big)$$

Proof. Let ℓ, b_1, b_2 and t. We assume that $\mathrm{LC}^{\ell}_{t}(b_1, b_2)$.

The proof is made by induction on d :

- In this case, $d = 0$.

  Because $\mathrm{LC}^{\ell}_{t}(b_1, b_2)$, we have that $\mathrm{Brd}^{\ell}_{t}(b_1)$ and $\mathrm{Brd}^{\ell}_{t}(b_2)$. So, by using the lemma B.6 we have that $\mathrm{dst}^{\ell}_{t}(b_1) = 0$ and $\mathrm{dst}^{\ell}_{t}(b_2) = 0$, and by definition (4) we have that $\mathrm{Sta}^{\ell}_{t}(b_1)$ and $\mathrm{Sta}^{\ell}_{t}(b_2)$.

  Moreover, for every $b_1 \leq c \leq b_2$ we have $\mathrm{dst}^{\ell}_{t}(c) \geq 0$ because dst is an integer field.

- We assume that $d + 1 \leq \frac{b_2 - b_1}{2}$. So $d \leq \frac{b_2 - b_1}{2}$ too, and we have the induction hypothesis:

$$\mathrm{dst}^{\ell}_{t+d}(b_1 + d) = d \wedge \mathrm{Sta}^{\ell}_{t+d}(b_1 + d)$$

$$\wedge\, \mathrm{dst}^{\ell}_{t+d}(b_2 - d) = d \wedge \mathrm{Sta}^{\ell}_{t+d}(b_2 - d)$$

$$\wedge \Big(\forall\ b_1 + d \leq c \leq b_2 - d, \mathrm{dst}^{\ell}_{t+d}(c) \geq d\Big)$$

Firstly, we prove that for every $b_1 + (d + 1) \leq c \leq b_2 - (d + 1)$, we have :

$$\mathrm{dst}^{\ell}_{t+(d+1)}(c) = 1 + \min\Big(\mathrm{dst}^{\ell}_{t+d}(c - 1), \mathrm{dst}^{\ell}_{t+d}(c + 1)\Big) \tag{H_c}$$

Indeed, if $b_1 + (d+1) \leq c \leq b_2 - (d+1)$ then by transitivity we have $b_1 < c < b_2$. So, because $\mathrm{LC}^{\ell}_{t}(b_1, b_2)$ we have $\mathrm{Ins}^{\ell}_{t+1}(c)$. So, by monotonicity (lemma C.10) we have $\mathrm{Ins}^{\ell}_{t+(d+1)}(c)$.

Therefore, by using the equation (6), we have H_c.

The proof is made by case on c :

- In that case, $c = b_1 + (d+1)$.

  Because $d + 1 \leq \frac{b_2 - b_1}{2}$, we have $2d + 2 \leq b_2 - b_1$, so $b_1 + d + 2 \leq b_2 - d$. So $b_1 + d \leq b_1 + d + 2 \leq b_2 - d$, and by using the induction hypothesis we have $\mathrm{dst}^{\ell}_{t+d}(b_1 + d + 2) \geq d$.

  Moreover, by using the induction hypothesis, we have $\mathrm{dst}^{\ell}_{t+d}(b_1 + d) = d$, so $\mathrm{dst}^{\ell}_{t+d}(b_1 + d + 2) \geq \mathrm{dst}^{\ell}_{t+d}(b_1 + d)$.

  By using H_c with $c = b_1 + d + 1$, we have :

$$\begin{aligned}\mathrm{dst}^{\ell}_{t+(d+1)}(b_1 + d + 1) &= 1 + \min\left(\mathrm{dst}^{\ell}_{t+d}(b_1 + d), \mathrm{dst}^{\ell}_{t+d}(b_1 + d + 2)\right)\\ &= 1 + \mathrm{dst}^{\ell}_{t+d}(b_1 + d)\\ &= 1 + d\end{aligned}$$

  Moreover, because $\mathrm{dst}^{\ell}_{t+(d+1)}(b_1 + d + 1) = 1 + \mathrm{dst}^{\ell}_{t+d}(b_1 + d)$ and by induction hypothesis $\mathrm{Sta}^{\ell}_{t+d}(b_1 + d)$, we have by definition (4) that $\mathrm{Sta}^{\ell}_{t+(d+1)}(b_1 + d + 1)$.

- The case $c = b_2 - (d+1)$ is similar, by using the induction hypothesis $\mathrm{dst}^{\ell}_{t+d}(b_2 - d) = d$ and $\mathrm{Sta}^{\ell}_{t+d}(b_2 - d)$.

- If $b_1 + (d+1) < c < b_2 - (d+1)$, then we have :

$$b_1 + d < c - 1 < b_2 - d - 2 < b_2 - d$$

$$b_1 + d < b_1 + d + 2 < c + 1 < b_2 - d$$

  So, by using the induction hypothesis we have $\mathrm{dst}^{\ell}_{t+d}(c-1) \geq d$ and $\mathrm{dst}^{\ell}_{t+d}(c+1) \geq d$.

  Therefore, by using H_c we have :

$$\begin{aligned}\mathrm{dst}^{\ell}_{t+(d+1)}(c) &= 1 + \min\left(\mathrm{dst}^{\ell}_{t+d}(c-1), \mathrm{dst}^{\ell}_{t+d}(c+1)\right)\\ &\geq 1 + \min(d, d)\\ &= 1 + d\end{aligned}$$

□

Corollary (5.5 p.12: Middle of an odd Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^{\ell}_t(b_1, b_2) \wedge b_2 - b_1 + 1 \,\mathrm{odd}$$

$$\Rightarrow \mathrm{Mid}^{\ell}_{t+\frac{b_2 - b_1}{2}}\left(\frac{b_1 + b_2}{2}\right)$$

Proof. Because $\mathrm{LC}^{\ell}_t(b_1, b_2)$ we have $b_1 + 2 \leq b_2$, so $\frac{b_2 - b_1}{2} \geq 1$.

Because $b_2 - b_1 + 1$ is odd, we have :

$$\frac{b_1 + b_2 + 1}{2} = \frac{b_1 + b_2}{2}$$

$$b_1 + \left(\frac{b_2 - b_1}{2} - 1\right) = \frac{b_1 + b_2}{2} - 1$$

$$b_1 - \left(\frac{b_2 - b_1}{2} - 1\right) = \frac{b_1 + b_2}{2} + 1$$

Because $\mathrm{LC}_t^\ell(b_1,b_2)$, by using the proposition 5.3 for $d=\frac{b_2-b_1}{2}-1$ we have:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}-1}(\frac{b_1+b_2}{2}-1)=\frac{b_2-b_1}{2}-1\wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}-1}(\frac{b_1+b_2}{2}-1)$$
$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}-1}(\frac{b_1+b_2}{2}+1)=\frac{b_2-b_1}{2}-1\wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}-1}(\frac{b_1+b_2}{2}+1)$$

Because $\mathrm{LC}_t^\ell(b_1,b_2)$, by using the proposition 5.3 for $d=\frac{b_2-b_1}{2}$ we have:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(\frac{b_1+b_2}{2})=\frac{b_2-b_1}{2}$$

So, by denoting $m=\frac{b_1+b_2}{2}$ we have :

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(m)>\max\left(\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}-1}(m-1),\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}-1}(m+1)\right)$$

with $\mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}-1}(m-1)$ and $\mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}-1}(m+1)$.

Therefore by definition (5) $\mathrm{Mid}^\ell_{t+\frac{b_2-b_1}{2}}(m)$. □

Corollary (5.6 p.13: Middles of an even Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}_t^\ell(b_1,b_2)\wedge b_2-b_1+1\,\text{even}$$
$$\Rightarrow \mathrm{Mid}^\ell_{t+\frac{b_2-b_1+1}{2}}\left(\frac{b_1+b_2-1}{2}\right)\wedge \mathrm{Mid}^\ell_{t+\frac{b_2-b_1+1}{2}}\left(\frac{b_1+b_2+1}{2}\right)$$

Proof. Because $\mathrm{LC}_t^\ell(b_1,b_2)$ we have $b_1+2\le b_2$, so because b_2-b_1 is odd we have $\frac{b_2-b_1-1}{2}\ge 1$.

Because b_2-b_1+1 is even, we have :

$$\frac{b_1+b_2+1}{2}=\frac{b_1+b_2-1}{2}+1$$
$$b_1+\left(\frac{b_2-b_1-1}{2}-1\right)=\frac{b_1+b_2-1}{2}-1$$
$$b_1-\left(\frac{b_2-b_1-1}{2}-1\right)=\frac{b_1+b_2+1}{2}+1$$

Because $\mathrm{LC}_t^\ell(b_1,b_2)$, by using the proposition 5.3 for $d=\frac{b_2-b_1-1}{2}-1$ we have:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1-1}{2}-1}(\frac{b_1+b_2-1}{2}-1)=\frac{b_2-b_1-1}{2}-1$$
$$\text{with }\ \mathrm{Sta}^\ell_{t+\frac{b_2-b_1-1}{2}-1}(\frac{b_1+b_2-1}{2}-1)$$
$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1-1}{2}-1}(\frac{b_1+b_2+1}{2}+1)=\frac{b_2-b_1-1}{2}-1$$
$$\text{with }\ \mathrm{Sta}^\ell_{t+\frac{b_2-b_1-1}{2}-1}(\frac{b_1+b_2+1}{2}+1)$$

So, by monotonicity (lemmas C.11 and C.14), we have :

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2-1}{2}-1)=\frac{b_2-b_1-1}{2}-1$$

$$\text{with } \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2-1}{2}-1)$$

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2+1}{2}+1) = \frac{b_2-b_1-1}{2}-1$$

$$\text{with } \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2+1}{2}+1)$$

Because $\mathrm{LC}^{\ell}_t(b_1,b_2)$, by using the proposition 5.3 for $d=\frac{b_2-b_1-1}{2}$ we have:

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2-1}{2}) = \frac{b_2-b_1-1}{2}$$

$$\text{with } \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2-1}{2})$$

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2+1}{2}) = \frac{b_2-b_1-1}{2}$$

$$\text{with } \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(\frac{b_1+b_2+1}{2})$$

Notice that $\frac{b_2-b_1-1}{2}+1=\frac{b_2-b_1+1}{2}$. So, by monotonicity (lemma C.14), we have :

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(\frac{b_1+b_2-1}{2}) = \frac{b_2-b_1-1}{2}$$

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(\frac{b_1+b_2+1}{2}) = \frac{b_2-b_1-1}{2}$$

So, by denoting $m_1=\frac{b_1+b_2-1}{2}$ and $m_2=\frac{b_1+b_2+1}{2}$ we have :

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(m_1) = \max\left(\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1-1), \mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1+1)\right)$$

with $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1-1)$, $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1)$ and $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_1+1)$.

Therefore by definition (5) $\mathrm{Mid}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(m_1)$.

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(m_2) = \max\left(\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2-1), \mathrm{dst}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2+1)\right)$$

with $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2-1)$, $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2)$ and $\mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1-1}{2}}(m_2+1)$.

Therefore by definition (5) $\mathrm{Mid}^{\ell}_{t+\frac{b_2-b_1+1}{2}}(m_2)$. □

Lemma (5.7 p.13: The other cells of a Light Cone are not Middles)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}^{\ell}_t(b_1,b_2) \Rightarrow \forall t' \geq t+\frac{b_2-b_1}{2}, \forall c,$$

$$\left(b_1 \leq c < \frac{b_1+b_2}{2} \vee \frac{b_1+b_2+1}{2} < c \leq b_2\right) \Rightarrow \neg\,\mathrm{Mid}^{\ell}_{t'+1}(c)$$

Proof. Let ℓ, t, b_1 and b_2. We assume the hypothesis $\mathrm{LC}^{\ell}_t(b_1,b_2)$.

Let $t' \geq t+\frac{b_2-b_1}{2}$ and let c be a cell.

Firstly, we prove that $c=b_1+d$ or $c=b_2-d$ with $0 \leq d < \frac{b_2-b_1}{2}$. The proof is made by case on c:

- If $b_1 \leq c < \frac{b_1+b_2}{2}$, $b_1 \leq c$ implies that $c = b_1 + d$. Moreover:

$$\text{We have } b_1 + d = c < \frac{b_1 + b_2}{2}$$

$$\text{Therefore } d < \frac{b_1 + b_2}{2} - b_1 = \frac{b_1 + b_2}{2} - \frac{2b_1}{2}$$

$$\text{Therefore (see the following remark) } d < \frac{b_1 + b_2 - 2b_1}{2} = \frac{b_2 - b_1}{2}$$

- If $\frac{b_1+b_2+1}{2} < c \leq b_2$, $c \leq b_2$ implies that $c = b_2 - d$. Moreover:

$$\text{We have } b_2 - d = c < \frac{b_1 + b_2 + 1}{2}$$

$$\text{Therefore } d < b_2 - \frac{b_1 + b_2 + 1}{2} = \frac{2b_2}{2} - \frac{b_1 + b_2 + 1}{2}$$

$$\text{Therefore (see the following remark) } d < \frac{2b_2 - b_1 - b_2}{2} = \frac{b_2 - b_1}{2}$$

By using the hypothesis $\mathrm{LC}_t^\ell(b_1, b_2)$ and the corollary 5.4 on $c = b_1 + d$ or $c = b_2 - d$, we have that:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(c) = d \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(c)$$

Therefore, because $t' + 1 \geq t' \geq t + \frac{b_2-b_1}{2}$, by using the lemma C.14 we have that:

$$\mathrm{dst}^\ell_{t'+1}(c) = d \tag{H_d}$$

Secondly, because $0 \leq d < \frac{b_2-b_1}{2}$, we have that $0 \leq d + 1 \leq \frac{b_2-b_1}{2}$.

So, by using the hypothesis $\mathrm{LC}_t^\ell(b_1, b_2)$ and the corollary 5.4 with $d + 1$, we have that:

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(b_1 + (d+1)) = d + 1 \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(b_1 + (d+1)) \tag{H_L}$$

$$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(b_2 - (d+1)) = d + 1 \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(b_2 - (d+1)) \tag{H_R}$$

We prove that $d + 1 \leq \max\left(\mathrm{dst}^\ell_{t'}(c-1), \mathrm{dst}^\ell_{t'}(c+1)\right)$ by case on c:

- If $b_1 \leq c < \frac{b_1+b_2}{2}$, we have $c = b_1 + d$, so $b_1 + (d+1) = c + 1$.
  Therefore, by using H_L, we have that:

  $$\mathrm{dst}^\ell_{t+\frac{b_2-b_1}{2}}(c+1) = d + 1 \wedge \mathrm{Sta}^\ell_{t+\frac{b_2-b_1}{2}}(c+1)$$

  So, because $t' \geq t + \frac{b_2-b_1}{2}$, by using the lemma C.14 we have that:

  $$\mathrm{dst}^\ell_{t'}(c+1) = d + 1$$

  Therefore: $d + 1 \leq \max\left(\mathrm{dst}^\ell_{t'}(c-1), \mathrm{dst}^\ell_{t'}(c+1)\right)$

- If $\frac{b_1+b_2+1}{2} < c \leq b_2$, we have $c = b_2 - d$, so $b_2 - (d+1) = c - 1$.

  Therefore, by using H_R, we have that:

$$\mathrm{dst}^{\ell}_{t+\frac{b_2-b_1}{2}}(c-1) = d+1 \wedge \mathrm{Sta}^{\ell}_{t+\frac{b_2-b_1}{2}}(c-1)$$

  So, because $t' \geq t + \frac{b_2-b_1}{2}$, by using the lemma C.14 we have that:

$$\mathrm{dst}^{\ell}_{t'}(c-1) = d+1$$

  Therefore: $d+1 \leq \max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big)$

We prove $\neg\,\mathrm{Mid}^{\ell}_{t'+1}(c)$ by contradiction. If $\mathrm{Mid}^{\ell}_{t'+1}(c)$ then, by using the lemma B.4, we have that:

$$\mathrm{dst}^{\ell}_{t'+1}(c) \geq \max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big)$$

So, by using H_d, we have that:

$$\max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big) \leq d$$

But $d+1 \leq \max\Big(\mathrm{dst}^{\ell}_{t'}(c-1), \mathrm{dst}^{\ell}_{t'}(c+1)\Big)$, hence the contradiction. □

Remark. In the previous lemma there is some parity problems to fix, which should be proven in the appendix or found in the Coq library.

Lemma (5.9 p.14: Each true Middle comes from a Light Cone)**.**

$$\forall \ell t m, \neg\,\mathrm{Brd}^{\ell}_{t}(m) \wedge \mathrm{Mid}^{\ell}_{t}(m)$$
$$\Rightarrow \mathrm{LC}^{\ell}_{t-d}(m-d, m+d)$$
$$\vee\, \mathrm{LC}^{\ell}_{t-(d+1)}(m-(d+1), m+d)$$
$$\vee\, \mathrm{LC}^{\ell}_{t-(d+1)}(m-d, m+(d+1))$$
$$\textit{where } d = \mathrm{dst}^{\ell}_{t}(m)$$

Proof. Let m be the "true" middle, which means that it is not a border. Notice that according to the lemma B.10 p.24, we have $d \geq 1$.

Firstly, we cannot have more than two true middles next to each other. Indeed, by (5) they must have the same distance d, but if there are three true middles in a row, the middle at the center will have a distance $d+1$, according to (6), hence the contradiction.

So, m may be alone, or have a middle with the same distance on its right or on its left, hence the three cases of the lemma.

Let m_ℓ the left middle and let m_r be the right middle. They are neighbors if there are two middles, or we have $m_\ell = m_r$ if there is only one middle.

Because m_ℓ and m_r are middles, according to the lemma B.8 p.23, they are stable.

In order to ease the notation, let t be the date when m_ℓ and m_r become stable for the first time. Therefore, by (5), if they are equal they become the middle at t, and otherwise they become the middles at $t+1$, hence the difference of time in the lemma.

Let ℓ be the layer. We prove by induction on $0 \leq i \leq d$ that:

$$\mathrm{Sta}^{\ell}_{t-i}(m_\ell - i) \wedge \mathrm{dst}^{\ell}_{t-i}(m_\ell - i) = d - i = \mathrm{dst}^{\ell}_{t-i}(m_r + i) \wedge \mathrm{Sta}^{\ell}_{t-i}(m_r + i)$$
$$\wedge\ \forall\ m_\ell - i \leq c \leq m_r + i, \mathrm{dst}^{\ell}_{t-i}(c) \geq d - i \qquad (H_i)$$

- H_0 because m_ℓ and m_r are stable at t and they have the same distance d.
- We assume H_i with $i < d$, and now we prove H_{i+1}.

  According to (4), we have $\mathrm{Sta}^\ell_{t-i}(m_\ell - i)$ because

  1. $\mathrm{Brd}^\ell_{t-i}(m_\ell - i)$
     In that case, according to the lemma B.6 p.23, we have $\mathrm{dst}^\ell_{t-i}(m_\ell - i) = 0$.
     So, by H_i we have d = i, which contradicts $i < d$.
  2. or $\mathrm{dst}^\ell_{t-i}(m_\ell - i) = 1 + \mathrm{dst}^\ell_{t-(i+1)}(m_\ell - i + 1)$ with $\mathrm{Sta}^\ell_{t-(i+1)}(m_\ell - i + 1)$
     In that case, because $\mathrm{Sta}^\ell_{t-(i+1)}(m_\ell - i + 1)$, according to the lemma C.14 p.32 we have $\mathrm{dst}^\ell_{t-(i+1)}(m_\ell - i + 1) = \mathrm{dst}^\ell_{t-i}(m_\ell - i + 1)$.
     But by H_i we have $\mathrm{dst}^\ell_{t-i}(m_\ell - i) = d - i$ and $\mathrm{dst}^\ell_{t-i}(m_\ell - i + 1) \geq d - i$.
     So, $d - i = \mathrm{dst}^\ell_{t-i}(m_\ell - i) = 1 + \mathrm{dst}^\ell_{t-(i+1)}(m_\ell - i + 1) \geq 1 + d - i$, which is absurd.
  3. or $\mathrm{dst}^\ell_{t-i}(m_\ell - i) = 1 + \mathrm{dst}^\ell_{t-(i+1)}(m_\ell - (i + 1))$ with $\mathrm{Sta}^\ell_{t-(i+1)}(m_\ell - (i + 1))$, which is the only remaining case.

  Therefore, we have $\mathrm{Sta}^\ell_{t-(i+1)}(m_\ell-(i+1))$, and $\mathrm{dst}^\ell_{t-(i+1)}(m_\ell-(i+1)) = \mathrm{dst}^\ell_{t-i}(m_\ell-i)-1 = d - (i + 1)$ by H_i.

  We prove $\mathrm{Sta}^\ell_{t-(i+1)}(m_r + (i + 1))$ and $\mathrm{dst}^\ell_{t-(i+1)}(m_r + (i + 1)) = d - (i + 1)$ in the same way, by using $\mathrm{Sta}^\ell_{t-i}(m_r + i)$ in H_i.

  It remains only to prove that for every $m_\ell-(i+1) \leq c \leq m_r+(i+1)$, we have $\mathrm{dst}^\ell_{t-(i+1)}(c) \geq d - (i + 1)$. We prove it by contradiction.

  Let c be a cell such that $m_\ell - (i + 1) \leq c \leq m_r + (i + 1)$ and $\mathrm{dst}^\ell_{t-(i+1)}(c) < d - (i + 1)$.

  There exists a cell c' which is a neighbor of c and $m_\ell - i \leq c' \leq m_r + i$. Indeed, if $m_\ell - i - 1 \leq c \leq m_\ell - i + 1$ let c' be $c + 1$, and if $m_\ell - i + 1 \leq c \leq m_r + i + 1$ let c' be $c - 1$.

  According to the lemma B.3 p.22, we have:

  $$\mathrm{dst}^\ell_{t-i+1}(c') \leq 1 + \mathrm{dst}^\ell_{t-(i+1)}(c) < 1 + d - (i + 1) = d - i$$

  which contradicts H_i.

Therefore, we proved H_i for every $0 \leq i \leq d$. We remind that $d \geq 1$, so we can apply H_i for $i = d - 1$ and $i = d$:

- H_d implies that $\mathrm{Sta}^\ell_{t-d}(m_\ell-d)$ and $\mathrm{dst}^\ell_{t-d}(m_\ell-d) = 0$ and $\mathrm{Sta}^\ell_{t-d}(m_r+d)$ and $\mathrm{dst}^\ell_{t-d}(m_r+d) = 0$, so according to the lemma B.9 p.24 we have $\mathrm{Brd}^\ell_{t-d}(m_\ell - d)$ and $\mathrm{Brd}^\ell_{t-d}(m_r + d)$.
- H_{d-1} implies that for every cell $m_\ell - d < c < m_r + d$ we have $\mathrm{dst}^\ell_{t-d+1}(c) \geq 1$, so by (3) we have $\mathrm{Ins}^\ell_{t-d+1}(c)$.

Moreover, because $d \geq 1$, we have that $m - d$, m, and $m + d$ are three different cells.
Therefore, by (7) we have $\mathrm{LC}^\ell_{t-d}(m_\ell - d, m_r + d)$. □

Corollary (5.10 p.14: A Middle induces a new Light Cone)**.**

$$\forall \ell t m d, \mathrm{Mid}^\ell_t(m) \wedge \mathrm{dst}^\ell_t(m) = d \wedge d \geq 2$$
$$\Rightarrow \forall t', \Big(\mathrm{Brd}^\ell_{t'}(m - d) \Rightarrow \mathrm{LC}^{\ell+1}_t(m - d, m)\Big)$$
$$\wedge \Big(\mathrm{Brd}^\ell_{t'}(m + d) \Rightarrow \mathrm{LC}^{\ell+1}_t(m, m + d)\Big)$$

Proof. Because $\mathrm{dst}_t^\ell(m) = d \geq 2$, $\mathrm{Mid}_t^\ell(m)$, by using the contraposition of the lemma B.6, we have that $\neg\,\mathrm{Brd}_t^\ell(m)$.

The proof is made by case on the border. We assume $\mathrm{Brd}_{t'}^\ell(m-d)$, but the proof in the case $\mathrm{Brd}_{t'}^\ell(m+d)$ is similar.

Because $\neg\,\mathrm{Brd}_t^\ell(m)$ and $\mathrm{Mid}_t^\ell(m)$, by using the previous lemma we have that $\mathrm{LC}_{t-d}^\ell(m-d, m+d)$ or $\mathrm{LC}_{t-(d+1)}^\ell(m-(d+1), m+d)$ or $\mathrm{LC}_{t-(d+1)}^\ell(m-d, m+(d+1))$.

By definition (7), if $\mathrm{LC}_{t-(d+1)}^\ell(m-(d+1), m+d)$ then $\mathrm{Ins}_{t-d}^\ell(m-d)$. So, by case $t' \leq t-d$ or $t-d \leq t'$ and by monotonicity, we obtain a contradiction dy using the lemma B.5. So we have :

$$\mathrm{LC}_{t-d}^\ell(m-d, m+d) \vee \mathrm{LC}_{t-(d+1)}^\ell(m-d, m+(d+1)) \qquad (H_{LC})$$

In every case, by using the lemma 5.4 we have for every $i \leq d$ that $\mathrm{dst}_t^\ell(m-d+i) = i$ and $\mathrm{Sta}_t^\ell(m-d+i)$.

So, by using the monotonicity, for every $m-d < c < m$, we have $\mathrm{dst}_{t+1}^\ell(c)+1 = \mathrm{dst}_t^\ell(c+1)$.

Moreover, by using the monotonicity, $\mathrm{Sta}_{t+1}^\ell(c)$.

Moreover, by using H_{LC} and the definition (7) and the monotonicity, we have $\mathrm{Ins}_{t+1}^\ell(c)$.

Therefore, by definition (3), we have $\mathrm{Ins}_{t+1}^{\ell+1}(c)$.

Moreover, by using H_{LC} and the definition (7) and the monotonicity, we have $\mathrm{Brd}_{t+1}^\ell(m-d)$. So, by definition (2) $\mathrm{Brd}_{t+1}^{\ell+1}(m-d)$.

Moreover, by using H_{LC} and (the lemma 5.5 or the lemma 5.6) and the monotonicity, we have $\mathrm{Mid}_{t+1}^\ell(m)$. So, by definition (2) $\mathrm{Brd}_{t+1}^{\ell+1}(m)$.

Moreover, by hypothesis $d \geq 2$, so $(m-d)+2 \leq m$.

Therefore, by definition (7) we have $\mathrm{LC}_t^{\ell+1}(m-d, m)$. ☐

The following lemma will ease the proof of the lemma 5.12 p.14 (and maybe others ?):

Lemma D.1 (Right Staircase Lemma)**.**

$$\forall \ell t c d, \mathrm{dst}_{t+2}^\ell(c) \geq d+1$$

$$\Rightarrow \mathrm{Sta}_{t+1}^\ell(c+1) \wedge \mathrm{dst}_{t+1}^\ell(c+1) = d+1$$

$$\Rightarrow \mathrm{Sta}_t^\ell(c+2) \wedge \mathrm{dst}_t^\ell(c+2) = d$$

Proof. Let ℓ, t, c and d.

We assume $\mathrm{dst}_{t+2}^\ell(c) \geq d+1$, $\mathrm{Sta}_{t+1}^\ell(c+1)$ and $\mathrm{dst}_{t+1}^\ell(c+1) = d+1$.

Because $\mathrm{Sta}_{t+1}^\ell(c+1)$, by (4) we have three possible cases:

- $\mathrm{Brd}_{t+1}^\ell(c+1)$

  So, according to the lemma B.6 p.23, we have $\mathrm{dst}_{t+1}^\ell(c+1) = 0$, which contradicts the hypothesis $\mathrm{dst}_{t+1}^\ell(c+1) = d+1$.

- $\mathrm{dst}_{t+1}^\ell(c+1) = 1 + \mathrm{dst}_t^\ell(c)$ with $\mathrm{Sta}_t^\ell(c)$

  Because $\mathrm{Sta}_t^\ell(c)$, according to the lemma C.14 p.32 we have $\mathrm{dst}_t^\ell(c) = \mathrm{dst}_{t+2}^\ell(c) \geq d+1$.

  Therefore $d+1 = \mathrm{dst}_{t+1}^\ell(c+1) = 1 + \mathrm{dst}_t^\ell(c) \geq d+2$, hence the contradiction.

- $\mathrm{dst}_{t+1}^\ell(c+1) = 1 + \mathrm{dst}_t^\ell(c+2)$ with $\mathrm{Sta}_t^\ell(c+2)$

  which is the only remaining case.

Therefore, $\mathrm{Sta}_t^\ell(c+2)$, and $\mathrm{dst}_t^\ell(c+2) = \mathrm{dst}_{t+1}^\ell(c+1) - 1 = d$. $\square$

The result in the other way (from bottom-right to top-left) is admitted, because the proof is similar.

Lemma (5.12 p.14: One Brd and one Mid at the previous layer of a Light Cone)**.**

$$\forall \ell t b_1 b_2, \mathrm{LC}_t^{\ell+1}(b_1, b_2)$$
$$\Rightarrow \Big(\mathrm{Brd}_t^\ell(b_1) \wedge \neg\,\mathrm{Brd}_t^\ell(b_2) \wedge \mathrm{Mid}_t^\ell(b_2) \wedge \mathrm{dst}_t^\ell(b_2) = b_2 - b_1\Big)$$
$$\vee \Big(\neg\,\mathrm{Brd}_t^\ell(b_1) \wedge \mathrm{Mid}_t^\ell(b_1) \wedge \mathrm{Brd}_t^\ell(b_2) \wedge \mathrm{dst}_t^\ell(b_1) = b_2 - b_1\Big)$$

Proof. Because $\mathrm{LC}_t^{\ell+1}(b_1, b_2)$, we have (7) that $b_1 + 2 \leq b_2$.

Because $\mathrm{LC}_t^{\ell+1}(b_1, b_2)$, we have (7) that $b_1 + 2 \leq b_2$ and for every cell $b_1 < c < b_2$ that $\mathrm{Ins}_{t+1}^{\ell+1}(c)$, so by (3) we have $\mathrm{Ins}_{t+1}^\ell(c)$.

Because $\mathrm{LC}_t^{\ell+1}(b_1, b_2)$, we have $\mathrm{Brd}_t^{\ell+1}(b_1)$ and $\mathrm{Brd}_t^{\ell+1}(b_2)$. So, by (2) and the lemma B.2 p.21, at the layer ℓ there is four possible cases :

1. $\mathrm{Brd}_t^\ell(b_1)$ and $\mathrm{Brd}_t^\ell(b_2)$

   In that case, because $b_1 + 2 \leq b_2$ and for every cell $b_1 < c < b_2$ we have $\mathrm{Ins}_{t+1}^\ell(c)$, we have $\mathrm{LC}_t^\ell(b_1, b_2)$.

   So, according to the corollary 5.5 p.12 or 5.6 p.13, we have $\mathrm{Mid}_{t+\frac{b_2-b_1+1}{2}}^\ell\left(\frac{b_1+b_2}{2}\right)$.

   Therefore (2) we have $\mathrm{Brd}_{t+\frac{b_2-b_1+1}{2}}^{\ell+1}\left(\frac{b_1+b_2}{2}\right)$.

   But $\mathrm{Ins}_{t+1}^{\ell+1}\left(\frac{b_1+b_2}{2}\right)$, so by monotony (lemma C.10 p.32) we have $\mathrm{Ins}_{t+\frac{b_2-b_1+1}{2}}^{\ell+1}\left(\frac{b_1+b_2}{2}\right)$.

   So, according to the lemma B.5 p.22, we have a contradiction.

2. $\neg\,\mathrm{Brd}_t^\ell(b_1) \wedge \mathrm{Mid}_t^\ell(b_1)$ and $\neg\,\mathrm{Brd}_t^\ell(b_2) \wedge \mathrm{Mid}_t^\ell(b_2)$

3. $\mathrm{Brd}_t^\ell(b_1)$ and $\neg\,\mathrm{Brd}_t^\ell(b_2) \wedge \mathrm{Mid}_t^\ell(b_2)$

4. $\neg\,\mathrm{Brd}_t^\ell(b_1) \wedge \mathrm{Mid}_t^\ell(b_1)$ and $\mathrm{Brd}_t^\ell(b_2)$

Let $d_1 = \mathrm{dst}_t^\ell(b_1)$. We assume for the moment that $\neg\,\mathrm{Brd}_t^\ell(b_1) \wedge \mathrm{Mid}_t^\ell(b_1)$ so, according to the lemma B.10 p.24, we have $d_1 \geq 1$. We prove by induction on $0 \leq i \leq d_1$ that:

$$\mathrm{Sta}_{t-i}^\ell(b_1 + i) \wedge \mathrm{dst}_{t-i}^\ell(b_1 + i) = d_1 - i \qquad (H_i)$$

- Because $\mathrm{Mid}_t^\ell(b_1)$, according to the lemma B.8 p.23, we have $\mathrm{Sta}_t^\ell(b_1)$.

  Moreover, $\mathrm{dst}_t^\ell(b_1) = d_1$, hence H_0.

- Because $\mathrm{Mid}_t^\ell(b_1)$, by (5) we have that $\mathrm{Sta}_{t-1}^\ell(b_1+1)$ and $\mathrm{dst}_{t-1}^\ell(b_1+1) = d_1 - 1$ or d_1.

  If $\mathrm{dst}_{t-1}^\ell(b_1+1) = d_1$, then according to the lemma D.1 p.40 we have $\mathrm{Sta}_{t-2}^\ell(b_1+2)$ and $\mathrm{dst}_{t-2}^\ell(b_1+2) = d_1 - 1$.

  So, by monotony (lemmas C.11 p.32 and C.14 p.32) we have $\mathrm{Sta}_t^\ell(b_1)$, $\mathrm{Sta}_t^\ell(b_1+1)$ and $\mathrm{Sta}_t^\ell(b_1+2)$ with $\mathrm{dst}_t^\ell(b_1) = d_1 = \mathrm{dst}_t^\ell(b_1+1)$ and $\mathrm{dst}_t^\ell(b_1+2) = d_1 - 1$.

  Therefore (5) $\mathrm{Mid}_{t+1}^\ell(b_1+1)$. So (2) $\mathrm{Brd}_{t+1}^{\ell+1}(b_1+1)$, which contradicts $\mathrm{Ins}_{t+1}^{\ell+1}(b_1+1)$, according to the lemma B.5 p.22.

  So $\mathrm{dst}_{t-1}^\ell(b_1+1) = d_1 - 1$, hence H_1.

- We assume H_i and H_{i+1} for $i \leq d_1 - 2$ and prove H_{i+2}.

  Because H_i we have $\mathrm{dst}^{\ell}_{t-i}(b_1 + i) = d_1 - i$.

  Because H_{i+1} we have $\mathrm{Sta}^{\ell}_{t-i-1}(b_1 + i + 1)$ and $\mathrm{dst}^{\ell}_{t-i-1}(b_1 + i + 1) = d_1 - i - 1$.

  According to the lemma D.1 p.40, we have $\mathrm{Sta}^{\ell}_{t-i-2}(b_1 + i + 2)$ and $\mathrm{dst}^{\ell}_{t-i-2}(b_1 + i + 2) = d_1 - i - 2$, hence H_{i+2}.

Let $d_2 = \mathrm{dst}^{\ell}_{t}(b_2)$. In the same way, if $\neg\,\mathrm{Brd}^{\ell}_{t}(b_2) \wedge \mathrm{Mid}^{\ell}_{t}(b_2)$ then we prove by induction on $0 \leq i \leq d_2$ that:

$$\mathrm{Sta}^{\ell}_{t-i}(b_2 - i) \wedge \mathrm{dst}^{\ell}_{t-i}(b_2 - i) = d_2 - i \qquad (H'_i)$$

If $\neg\,\mathrm{Brd}^{\ell}_{t}(b_1) \wedge \mathrm{Mid}^{\ell}_{t}(b_1)$, according to H_{d_1} we have $\mathrm{Sta}^{\ell}_{t-d_1}(b_1 + d_1)$ and $\mathrm{dst}^{\ell}_{t-d_1}(b_1 + d_1) = 0$.
So, according to the lemma B.9 p.24 we have $\mathrm{Brd}^{\ell}_{t-d_1}(b_1 + d_1)$.
So, by monotony (lemma C.9 p.31) we have $\mathrm{Brd}^{\ell}_{t+1}(b_1 + d_1)$.
If $d_1 < b_2 - b_1$ then $\mathrm{Ins}^{\ell}_{t+1}(b_1 + d_1)$, which contradicts $\mathrm{Brd}^{\ell}_{t+1}(b_1 + d_1)$, according to the lemma B.10 p.24.
Therefore we have $b_2 - b_1 \leq d_1$.
In the same way, if $\neg\,\mathrm{Brd}^{\ell}_{t}(b_2) \wedge \mathrm{Mid}^{\ell}_{t}(b_2)$, we use H'_{d_2} to prove that $b_2 - b_1 \leq d_2$.
W can go back to our remaining cases:

2. In the second case, where both b_1 and b_2 are true middles, we have:

   - According to $H_{b_2-b_1}$, we have $\mathrm{Sta}^{\ell}_{t-(b_2-b_1)}(b_2)$ and $\mathrm{dst}^{\ell}_{t-(b_2-b_1)}(b_2) = d_1 - (b_2 - b_1)$, therefore by monotony $d_2 = \mathrm{dst}^{\ell}_{t}(b_2) = d_1 - (b_2 - b_1) < d_1$.
   - According to $H'_{b_2-b_1}$, we have $\mathrm{Sta}^{\ell}_{t-(b_2-b_1)}(b_1)$ and $\mathrm{dst}^{\ell}_{t-(b_2-b_1)}(b_1) = d_2 - (b_2 - b_1)$, therefore by monotony $d_1 = \mathrm{dst}^{\ell}_{t}(b_1) = d_2 - (b_2 - b_1) < d_2$.

   Hence the contradiction.

3. In the third case, because $\mathrm{Brd}^{\ell}_{t}(b_1)$ according to the lemma B.6 p.23 we have $d_1 = 0$.

   Because $\neg\,\mathrm{Brd}^{\ell}_{t}(b_2) \wedge \mathrm{Mid}^{\ell}_{t}(b_2)$, we use $H'_{b_2-b_1}$ to prove that $d_1 = d_2 - (b_2 - b_1)$

   Therefore $d_2 = b_2 - b_1$.

4. In the fourth case, because $\mathrm{Brd}^{\ell}_{t}(b_2)$ according to the lemma B.6 p.23 we have $d_2 = 0$.

   Because $\neg\,\mathrm{Brd}^{\ell}_{t}(b_1) \wedge \mathrm{Mid}^{\ell}_{t}(b_1)$, we use $H_{b_2-b_1}$ to prove that $d_2 = d_1 - (b_2 - b_1)$.

   Therefore $d_1 = b_2 - b_1$.

□

E Proofs for the Middles

The proofs of the following results are very detailed and written in a Coq style, in order to be implemented. Therefore, for sake of clarity, we will only write and comment the results in the section 6 p.15, and detail the proofs in this section.

Lemma (6.1 p.15: Paired Middles appear at the same time with the same distance)**.**

$$\forall \ell t_1 t_2 m_1 m_2, \mathrm{Mid}^{\ell}_{t_1}(m_1) \wedge \mathrm{Mid}^{\ell}_{t_2}(m_2) \wedge (m_2 = m_1 + 1 \vee m_1 = m_2 + 1)$$

$$\Rightarrow \mathrm{Mid}^{\ell}_{t_1}(m_2) \wedge \mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)$$

Proof. We assume that $m_2 = m_1 + 1$ (the case $m_1 = m_2 + 1$ is symmetrical). For sake of simplicity, we note $c_1 = m_1 - 1$ and $c_2 = m_2 + 1$.

We assume that $t_1 \leq t_2$ and we prove the result both for t_1 and t_2. Notice that because of the middles, we have $t_1, t_2 \geq 1$.

We note $d_1 = \mathrm{dst}^{\ell}_{t_1}(m_1)$ and $d_2 = \mathrm{dst}^{\ell}_{t_2}(m_2)$, and we prove that $d_1 = d_2$:

According to the lemma B.4 on $\mathrm{Mid}^{\ell}_{t_1}(m_1)$ we have that :

$$d_1 \geq \max\left(\mathrm{dst}^{\ell}_{t_1-1}(c_1), \mathrm{dst}^{\ell}_{t_1-1}(m_2)\right) \geq \mathrm{dst}^{\ell}_{t_1-1}(m_2)$$

According to the lemma B.7 on $\mathrm{Mid}^{\ell}_{t_1}(m_1)$, we have $\mathrm{Sta}^{\ell}_{t_1-1}(m_2)$. So, by monotonicity (lemma C.14), $\mathrm{dst}^{\ell}_{t_1-1}(m_2) = d_2$. Therefore $d_1 \geq d_2$.

According to the lemma B.4 on $\mathrm{Mid}^{\ell}_{t_2}(m_2)$ we have that :

$$d_2 \geq \max\left(\mathrm{dst}^{\ell}_{t_2-1}(m_1), \mathrm{dst}^{\ell}_{t_2-1}(c_2)\right) \geq \mathrm{dst}^{\ell}_{t_2-1}(m_1)$$

We have two cases on $t_1 \leq t_2$:

- In the case $t_1 = t_2$, according to the lemma B.7 on $\mathrm{Mid}^{\ell}_{t_2}(m_2)$, we have $\mathrm{Sta}^{\ell}_{t_2-1}(m_1)$. So, by monotonicity (lemma C.14), $\mathrm{dst}^{\ell}_{t_2-1}(m_1) = d_1$.
- In the case $t_1 < t_2$, according to the lemma B.8 on $\mathrm{Mid}^{\ell}_{t_1}(m_1)$ we have $\mathrm{Sta}^{\ell}_{t_1}(m_1)$. So, by monotonicity (lemma C.14), $\mathrm{dst}^{\ell}_{t_2-1}(m_1) = d_1$.

In every case, we have $d_2 \geq d_1$, and because we proved $d_1 \geq d_2$, we have $d_1 = d_2$. So, in the following d_1 and d_2 will be denoted by d.

Because $\mathrm{dst}^{\ell}_{t_1}(m_1) = d = \mathrm{dst}^{\ell}_{t_1-1}(m_2)$, the middle m_1 verifies the second case of the equation (5). In particular, we have that $\mathrm{Sta}^{\ell}_{t_1-1}(m_1)$. So, by monotonicity (lemma C.14) we have $\mathrm{dst}^{\ell}_{t_1-1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_1) = d$.

We have two cases on d :

- If $\mathrm{dst}^{\ell}_{t_1-1}(m_2) = d = 0$, because $\mathrm{Sta}^{\ell}_{t_1-1}(m_2)$, by (4) we have $\mathrm{Brd}^{\ell}_{t_1-1}(m_2)$... [2]
- If $\mathrm{dst}^{\ell}_{t_1-1}(m_2) = d > 0$ [3], because $\mathrm{Sta}^{\ell}_{t_1-1}(m_2)$, by (4) we have two cases:
  - $\mathrm{dst}^{\ell}_{t_1-1}(m_2) = 1 + \mathrm{dst}^{\ell}_{t_1-2}(m_1) \wedge \mathrm{Sta}^{\ell}_{t_1-2}(m_1)$
    In that case, because $\mathrm{Sta}^{\ell}_{t_1-2}(m_1)$, by monotonicity (lemma C.14 p.32) we have that $\mathrm{dst}^{\ell}_{t_1-2}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_1) = d$.
    Therefore $d = \mathrm{dst}^{\ell}_{t_1-1}(m_2) = 1 + \mathrm{dst}^{\ell}_{t_1-2}(m_1) = 1 + d$, hence the contradiction.
  - $\mathrm{dst}^{\ell}_{t_1-1}(m_2) = 1 + \mathrm{dst}^{\ell}_{t_1-2}(c_2) \wedge \mathrm{Sta}^{\ell}_{t_1-2}(c_2)$.
    So $\mathrm{dst}^{\ell}_{t_1-2}(c_2) = d-1$, and by monotonicity (lemma C.14) we have $\mathrm{dst}^{\ell}_{t_1-1}(c_2) = d-1$.
    Moreover, because $\mathrm{Sta}^{\ell}_{t_1-2}(c_2)$, by monotonicity (lemma C.11) we have $\mathrm{Sta}^{\ell}_{t_1-1}(c_2)$.
    Finally, because $\mathrm{dst}^{\ell}_{t_1-1}(m_2) = d_2$ and $\mathrm{Sta}^{\ell}_{t_1-1}(m_2)$, by monotonicity (lemma C.14) we have $\mathrm{dst}^{\ell}_{t_1}(m_2) = d = \mathrm{dst}^{\ell}_{t_1}(m_1)$.
    Therefore, we have :

[2] The proof is not finished, but this case may not be necessary, because it cannot happen in the case $\ell = 0$ by axiom $n > 2$ and the definition (2) of Brd, and this lemma is only used in that case.

[3] In that case, because the distance is 0 at $t = 0$, we have $t_1 - 1 > 0$, so we can write $t_1 - 2$.

* $\mathrm{dst}^{\ell}_{t_1-1}(m_1) = d$ and $\mathrm{dst}^{\ell}_{t_1-1}(c_2) = d-1$, so :

$$\mathrm{dst}^{\ell}_{t_1}(m_2) = d = \max(d, d-1) = \max\left(\mathrm{dst}^{\ell}_{t_1-1}(m_1), \mathrm{dst}^{\ell}_{t_1-1}(c_2)\right)$$

* $\mathrm{Sta}^{\ell}_{t_1-1}(m_1)$ and $\mathrm{Sta}^{\ell}_{t_1-1}(m_2)$ and $\mathrm{Sta}^{\ell}_{t_1-1}(c_2)$

So, by the definition (5), we have $\mathrm{Mid}^{\ell}_{t_1}(m_2)$.

The result can be proven for t_2 too according to the monotonicity. □

Proposition (6.3 p.15: Middles appear at the same time with the same distance)**.**

$$\forall \ell t_1 m_1, \neg\,\mathrm{Brd}^{\ell}_{t_1}(m_1) \wedge \mathrm{Mid}^{\ell}_{t_1}(m_1)$$

$$\Rightarrow \left(\forall t_2 m_2, \mathrm{Mid}^{\ell}_{t_2}(m_2) \Rightarrow \mathrm{Mid}^{\ell}_{t_1}(m_2) \wedge \mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)\right)$$

Proof. The proof is made by induction on ℓ:

- In this case $\ell = 0$.
  Let t_1 and m_1 such that $\neg\,\mathrm{Brd}^{0}_{t_1}(m_1)$ and $\mathrm{Mid}^{0}_{t_1}(m_1)$.
  Let t_2 and m_2 such that $\mathrm{Mid}^{0}_{t_2}(m_2)$.
  Because $\ell = 0$, according to the lemma 5.2 there exists t_{LC} such that $\mathrm{LC}^{0}_{t_{\mathrm{LC}}}(1, n)$. We prove $\mathrm{Mid}^{\ell}_{t_1}(m_2)$ and $\mathrm{dst}^{\ell}_{t_1}(m_1) = \mathrm{dst}^{\ell}_{t_1}(m_2)$ by case on the parity of n:
  - If $n = n-1+1$ is odd, then according to the corollary 5.5 we have $\mathrm{Mid}^{0}_{t_{\mathrm{LC}}+\frac{n-1}{2}}(\frac{n+1}{2})$.
    By contradiction, if $m_1 \neq \frac{n+1}{2}$, then according to the lemma 5.8 we have that $\neg\,\mathrm{Mid}^{0}_{t_1}(m_1)$, which contradicts the hypothesis $\mathrm{Mid}^{0}_{t_1}(m_1)$. So $m_1 = \frac{n+1}{2}$.
    By contradiction, if $m_2 \neq \frac{n+1}{2}$, then according to the lemma 5.8 we have that $\neg\,\mathrm{Mid}^{0}_{t_2}(m_2)$, which contradicts the hypothesis $\mathrm{Mid}^{0}_{t_2}(m_2)$. So $m_2 = \frac{n+1}{2}$.
    Therefore, $m_1 = m_2$, then by hypothesis $\mathrm{Mid}^{0}_{t_1}(m_2)$, and we have $\mathrm{dst}^{0}_{t_1}(m_1) = \mathrm{dst}^{0}_{t_1}(m_2)$.
  - If $n = n-1+1$ is even, then according to the corollary 5.6 we have that $\mathrm{Mid}^{0}_{t_{\mathrm{LC}}+\frac{n}{2}}(\frac{n}{2})$ and $\mathrm{Mid}^{0}_{t_{\mathrm{LC}}+\frac{n}{2}}(\frac{n}{2}+1)$.
    By contradiction, if $m_1 \neq \frac{n}{2}$ and $m_1 \neq \frac{n}{2}+1$, then according to the lemma 5.8 we have that $\neg\,\mathrm{Mid}^{0}_{t_1}(m_1)$, which contradicts the hypothesis $\mathrm{Mid}^{0}_{t_1}(m_1)$. So $m_1 = \frac{n}{2}$ or $m_1 = \frac{n}{2}+1$.
    By contradiction, if $m_2 \neq \frac{n}{2}$ and $m_2 \neq \frac{n}{2}+1$, then according to the lemma 5.8 we have that $\neg\,\mathrm{Mid}^{0}_{t_2}(m_2)$, which contradicts the hypothesis $\mathrm{Mid}^{0}_{t_2}(m_2)$. So $m_2 = \frac{n}{2}$ or $m_2 = \frac{n}{2}+1$.
    The proof is made by case:
    * If $m_1 = m_2$, then by hypothesis $\mathrm{Mid}^{0}_{t_1}(m_2)$, and we have $\mathrm{dst}^{0}_{t_1}(m_1) = \mathrm{dst}^{0}_{t_1}(m_2)$.
    * If $m_1 \neq m_2$, then $m_2 = m_1 + 1$ or $m_1 = m_2 + 1$. So, because $\mathrm{Mid}^{0}_{t_1}(m_1)$ and $\mathrm{Mid}^{0}_{t_2}(m_2)$, according to the lemma 6.1 we have $\mathrm{Mid}^{0}_{t_1}(m_2)$ and $\mathrm{dst}^{0}_{t_1}(m_1) = \mathrm{dst}^{0}_{t_1}(m_2)$.

- We assume the induction hypothesis:

$$\forall t_1 m_1, \neg\operatorname{Brd}^{\ell}_{t_1}(m_1) \wedge \operatorname{Mid}^{\ell}_{t_1}(m_1) \tag{IH_ℓ}$$

$$\Rightarrow \Big(\forall t_2 m_2, \operatorname{Mid}^{\ell}_{t_2}(m_2) \Rightarrow \operatorname{Mid}^{\ell}_{t_1}(m_2) \wedge \operatorname{dst}^{\ell}_{t_1}(m_1) = \operatorname{dst}^{\ell}_{t_1}(m_2)\Big)$$

Let t_1 and m_1 such that $\neg\operatorname{Brd}^{\ell+1}_{t_1}(m_1)$ and $\operatorname{Mid}^{\ell+1}_{t_1}(m_1)$, and let $d_1 = \operatorname{dst}^{\ell+1}_{t_1}(m_1)$.

According to the lemma 5.9, there exists $t_1' = t_1 - d_1$ or $t_1 - (d_1+1)$, $b_1 = m_1 - d_1$ or $m_1 - (d_1+1)$, and $b_1' = m_1 + d_1$ or $m_1 + (d_1+1)$ such that $\operatorname{LC}^{\ell+1}_{t_1'}(b_1, b_1')$.

Notice that the case $b_1 = m_1 - (d_1+1)$ and $b_1' = m_1 + (d_1+1)$ is excluded, so $b_1' - b_1 = 2d_1$ or $2d_1+1$, but not $2d_1+2$.

Because $\operatorname{LC}^{\ell+1}_{t_1'}(b_1, b_1')$, according to the lemma 5.12, we have that:

$$\text{either}\quad \operatorname{Brd}^{\ell}_{t_1'}(b_1) \wedge \neg\operatorname{Brd}^{\ell}_{t_1'}(b_1') \wedge \operatorname{Mid}^{\ell}_{t_1'}(b_1') \wedge \operatorname{dst}^{\ell}_{t_1'}(b_1') = b_1' - b_1$$

$$\text{or}\quad \neg\operatorname{Brd}^{\ell}_{t_1'}(b_1) \wedge \operatorname{Mid}^{\ell}_{t_1'}(b_1) \wedge \operatorname{Brd}^{\ell}_{t_1'}(b_1') \wedge \operatorname{dst}^{\ell}_{t_1'}(b_1) = b_1' - b_1$$

We denote the border by b_1^ℓ and the middle by m_1^ℓ. In particular, we have that $\operatorname{dst}^{\ell}_{t_1'}(m_1^\ell) = b_1' - b_1 = 2d_1$ or $2d_1+1$.

Let t_2 and m_2 such that $\operatorname{Mid}^{\ell+1}_{t_2}(m_2)$, and let $d_2 = \operatorname{dst}^{\ell+1}_{t_2}(m_2)$.

According to the same arguments, we have that $\operatorname{LC}^{\ell+1}_{t_2'}(b_2, b_2')$, and at the previous layer we denote the border by b_2^ℓ and the middle by m_2^ℓ, with $\operatorname{dst}^{\ell}_{t_2'}(m_2^\ell) = b_2' - b_2 = 2d_2$ or $2d_2+1$.

Because $\neg\operatorname{Brd}^{\ell}_{t_1'}(m_1^\ell)$ and $\operatorname{Mid}^{\ell}_{t_1'}(m_1^\ell)$ and $\operatorname{Mid}^{\ell}_{t_2'}(m_2^\ell)$, according to the induction hypothesis IH_ℓ, we have that $\operatorname{Mid}^{\ell}_{t_1'}(m_2^\ell)$ and $\operatorname{dst}^{\ell}_{t_1'}(m_1^\ell) = \operatorname{dst}^{\ell}_{t_1'}(m_2^\ell)$.

Because $\operatorname{Mid}^{\ell}_{t_1'}(m_2^\ell)$ and $\operatorname{Mid}^{\ell}_{t_2'}(m_2^\ell)$, according to the lemma 6.2 we have that $\operatorname{dst}^{\ell}_{t_1'}(m_2^\ell) = \operatorname{dst}^{\ell}_{t_2'}(m_2^\ell)$.

Therefore $\operatorname{dst}^{\ell}_{t_1'}(m_1^\ell) = \operatorname{dst}^{\ell}_{t_1'}(m_2^\ell) = \operatorname{dst}^{\ell}_{t_2'}(m_2^\ell)$.

So, because $\operatorname{dst}^{\ell}_{t_1'}(m_1^\ell) = 2d_1$ or $2d_1+1$ and $\operatorname{dst}^{\ell}_{t_2'}(m_2^\ell) = 2d_2$ or $2d_2+1$, according to the lemma B.1 we have that $d_1 = d_2$.

Therefore $\operatorname{dst}^{\ell+1}_{t_1}(m_1) = d_1 = d_2 = \operatorname{dst}^{\ell+1}_{t_2}(m_2)$.

It remains to prove that $\operatorname{Mid}^{\ell+1}_{t_1}(m_2)$.

Because $\neg\operatorname{Brd}^{\ell+1}_{t_1}(m_1)$ and $\operatorname{Mid}^{\ell+1}_{t_1}(m_1)$, according to the lemma B.10 we have that $d_2 = d_1 = \operatorname{dst}^{\ell+1}_{t_1}(m_1) \geq 1$.

So $\operatorname{dst}^{\ell}_{t_1'}(m_2^\ell) = \operatorname{dst}^{\ell}_{t_2'}(m_2^\ell) = 2d_2$ or $2d_2+1 \geq 2$.

Moreover, because $\operatorname{dst}^{\ell}_{t_1'}(m_2^\ell) = \operatorname{dst}^{\ell}_{t_2'}(m_2^\ell) = b_2' - b_2$, where b_2 and b_2' are b_2^ℓ and m_2^ℓ or the reverse, we have $b_2^\ell = m_2^\ell - \operatorname{dst}^{\ell}_{t_1'}(m_2^\ell)$ or $b_2^\ell = m_2^\ell + \operatorname{dst}^{\ell}_{t_1'}(m_2^\ell)$.

Moreover, $\operatorname{Mid}^{\ell}_{t_1'}(m_2^\ell)$ and $\operatorname{Brd}^{\ell}_{t_2'}(b_2^\ell)$.

Therefore, according to the lemma 5.10, we have $\operatorname{LC}^{\ell+1}_{t_1'}(b_2, b_2')$.

We prove $\operatorname{Mid}^{\ell+1}_{t_1}(m_2)$ by case on the parity of $b_2' - b_2$:

– If $b_2' - b_2$ is even, because $b_2' - b_2 = 2d_2$ or $2d_2 + 1$, we have $b_2' - b_2 = 2d_2$ so $\frac{b_2' - b_2}{2} = d_2$. Because $b_2' - b_2$ is even, $b_2' - b_2 + 1$ is odd. So, according to the lemma 5.5, we have that $\mathrm{Mid}^{\ell+1}_{t_1' + \frac{b_2' - b_2}{2}}(\frac{b_2 + b_2'}{2})$.

In that case (according to the previous results of the lemma 5.9), we have (see the following remark) $t_1' = t_1 - d_1$ and $b_2 = m_2 - d_2$ and $b_2' = m_2 + d_2$, so :

$$t_1' + \frac{b_2' - b_2}{2} = t_1' + d_2 = t_1' + d_1 = t_1$$

$$\frac{b_2 + b_2'}{2} = \frac{(m_2 - d_2) + (m_2 + d_2)}{2} = \frac{2m_2}{2} = m_2$$

Therefore $\mathrm{Mid}^{\ell+1}_{t_1}(m_2)$.

– If $b_2' - b_2$ is odd, because $b_2' - b_2 = 2d_2$ or $2d_2 + 1$, we have $b_2' - b_2 = 2d_2 + 1$ so $\frac{b_2' - b_2 + 1}{2} = d_2 + 1$.

Because $b_2' - b_2$ is odd, $b_2' - b_2 + 1$ is even. So, according to the lemma 5.6, we have that $\mathrm{Mid}^{\ell+1}_{t_1' + \frac{b_2' - b_2 + 1}{2}}(\frac{b_2 + b_2' - 1}{2})$ and $\mathrm{Mid}^{\ell+1}_{t_1' + \frac{b_2' - b_2 + 1}{2}}(\frac{b_2 + b_2' + 1}{2})$.

In that case (according to the previous results of the lemma 5.9), we have (see the following remark) $t_1' = t_1 - (d_1 + 1)$, so:

$$t_1' + \frac{b_2' - b_2 + 1}{2} = t_1' + d_2 + 1 = t_1' + d_1 + 1 = t_1$$

Morevover, there are two cases for b_2 and b_2':

 * $b_2 = m_2 - (d_2 + 1)$ and $b_2' = m_2 + d_2$. In that case:

$$\frac{b_2 + b_2' + 1}{2} = \frac{(m_2 - d_2 - 1) + (m_2 + d_2) + 1}{2} = \frac{2m_2}{2} = m_2$$

 Therefore, because $\mathrm{Mid}^{\ell+1}_{t_1' + \frac{b_2' - b_2 + 1}{2}}(\frac{b_2 + b_2' - 1}{2})$, we have that $\mathrm{Mid}^{\ell+1}_{t_1}(m_2)$.

 * $b_2 = m_2 - d_2$ and $b_2' = m_2 + (d_2 + 1)$. In that case:

$$\frac{b_2 + b_2' - 1}{2} = \frac{(m_2 - d_2) + (m_2 + d_2 + 1) - 1}{2} = \frac{2m_2}{2} = m_2$$

 Therefore, because $\mathrm{Mid}^{\ell+1}_{t_1' + \frac{b_2' - b_2 + 1}{2}}(\frac{b_2 + b_2' + 1}{2})$, we have that $\mathrm{Mid}^{\ell+1}_{t_1}(m_2)$.

□

Remark. The remaining problems in the proposition come from the fact that the cases for the form of the Light Cones "may" not be the same (in particular even or odd length) for the two middles. Maybe we should prove that this is the case anyway because at a layer ℓ the Light Cones have the same length ?

Lemma (6.4 p.16: Three true middles cannot be adjacent)**.**

$$\forall \ell t c, \neg\, \mathrm{Brd}^{\ell}_{t}(c) \wedge \mathrm{Mid}^{\ell}_{t}(c-1) \wedge \mathrm{Mid}^{\ell}_{t}(c) \wedge \mathrm{Mid}^{\ell}_{t}(c+1) \Rightarrow \text{False}$$

Proof. We obtain a contradiction by case on t:

- If $t = 0$ then (5) $\mathrm{Mid}_0^\ell(c)$ is False.
- Else $t = t' + 1$. By hypothesis $\neg\,\mathrm{Brd}_{t'+1}^\ell(c)$, so we can use the proposition 6.3 to prove that:

  $$\mathrm{dst}_{t'+1}^\ell(c-1) = \mathrm{dst}_{t'+1}^\ell(c) = \mathrm{dst}_{t'+1}^\ell(c+1)$$

  This distance will be denoted by d.

  By using the lemma B.7 on $\mathrm{Mid}_{t'+1}^\ell(c)$, we have that $\mathrm{Sta}_{t'}^\ell(c-1)$ and $\mathrm{Sta}_{t'}^\ell(c+1)$. So, by using the lemma C.14 on both we have:

  $$\mathrm{dst}_{t'}^\ell(c-1) = \mathrm{dst}_{t'+1}^\ell(c-1) = d$$

  $$\mathrm{dst}_{t'}^\ell(c+1) = \mathrm{dst}_{t'+1}^\ell(c+1) = d$$

  Because $\neg\,\mathrm{Brd}_{t'+1}^\ell(c)$ and $\mathrm{Mid}_{t'+1}^\ell(c)$, by using the lemma B.10 we have $\mathrm{dst}_{t'+1}^\ell(c) > 0$. So (6):

  $$\begin{aligned}\mathrm{dst}_{t'+1}^\ell(c) &= 1 + \min\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right)\\ &= 1 + \min(d, d)\\ &= 1 + d\end{aligned}$$

  which contradicts $\mathrm{dst}_{t'+1}^\ell(c) = d$.

□

Lemma (6.5 p.16: A true middle adjacent to a border has a distance = 1)**.**

$$\forall \ell t c, \neg\,\mathrm{Brd}_t^\ell(c) \wedge \mathrm{Mid}_t^\ell(c) \wedge \left(\mathrm{Brd}_t^\ell(c-1) \vee \mathrm{Brd}_t^\ell(c+1)\right) \Rightarrow \mathrm{dst}_t^\ell(c) = 1$$

Proof. We prove the result by case on t:

- If $t = 0$ then (5) $\mathrm{Mid}_0^\ell(c)$ is False, so we get a contradiction.
- Else $t = t' + 1$. Because $\neg\,\mathrm{Brd}_{t'+1}^\ell(c)$ and $\mathrm{Mid}_{t'+1}^\ell(c)$, by using the lemma B.10 we have $\mathrm{dst}_{t'+1}^\ell(c) > 0$. So (6):

  $$\mathrm{dst}_{t'+1}^\ell(c) = 1 + \min\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right)$$

  But $\left(\mathrm{Brd}_{t'+1}^\ell(c-1) \vee \mathrm{Brd}_{t'+1}^\ell(c+1)\right)$.

  So by using the lemma B.6 we have $\left(\mathrm{dst}_{t'+1}^\ell(c-1) = 0 \vee \mathrm{dst}_{t'+1}^\ell(c+1) = 0\right)$, and by using the lemma C.13 we have $\left(\mathrm{dst}_{t'}^\ell(c-1) = 0 \vee \mathrm{dst}_{t'}^\ell(c+1) = 0\right)$.

  Therefore, $\min\left(\mathrm{dst}_{t'}^\ell(c-1), \mathrm{dst}_{t'}^\ell(c+1)\right) = 0$, and $\mathrm{dst}_{t'+1}^\ell(c) = 1$.

□

F Implementation in Coq

We began to write our presentation in Coq, but most of the work remains to be done. At least, this technical report attests the faisability of the task. For the moment, we wrote in the following Coq code:

- the definitions (from p.5 to p.8), both in boolean and propositional form, of the fields Inp, Brd, Ins, Sta, Mid, and dst
- the proof of the lemmas 3.4 p.5 and 3.5 p.6, stating the equivalence between the boolean and the propositional forms
- the proof of the lemma B.3 p.22 (Local Distance)
- the proof of the lemma C.2 p.25 (Ins monotone implies dst is increasing)
- the proof of the lemma B.5 p.22 (Brd and Ins are exclusive)
- the proof of the lemma B.6 p.23 (Distance of a Border)
- the proof of the lemma C.3 p.25 (Brd and Ins monotone implies a stable dst is constant)

```
Require Import Bool.
Require Import Coq.Arith.Compare_dec.
Require Import Coq.Arith.Max.
Require Import Coq.Arith.Min.
Require Import Coq.Arith.Lt.
Require Import Coq.Arith.PeanoNat.
Require Import Coq.Init.Nat.

(* Technical lemmas *)

Section le_min_max.

  Lemma min_or :
    forall m n, min m n = m ∨ min m n = n.

    Proof. intros m n. destruct (Nat.le_ge_cases m n).
    - left. apply min_l. apply H.
    - right. apply min_r. apply H.
    Qed.

  Lemma min_le_min :
    forall m1 m2 n1 n2, m1 <= m2 → n1 <= n2 → min m1 n1 <= min m2 n2.

    Proof. intros m1 m2 n1 n2 Hm Hn. destruct (min_or m2 n2).
    - rewrite → H. transitivity m1. apply le_min_l. apply Hm.
    - rewrite → H. transitivity n1. apply le_min_r. apply Hn.
    Qed.

  Lemma max_le__and_le :
    forall n1 n2 m, max n1 n2 <= m → (n1 <= m ∧ n2 <= m).

  Proof. split.
```

```
  - transitivity (max n1 n2). apply le_max_l. apply H.
  - transitivity (max n1 n2). apply le_max_r. apply H.
  Qed.

End le_min_max.

(* Most field F have the form: F l t c
   where l is the layer/level, t the time, and c the cell/position *)

Definition Level := nat.
Definition Time := nat.
Definition Space := nat.

Section Evolution.

(* An evolution is given with the starting position(s) of the general(s)
   and the number n of cells *)

  Variable Evo : Set.
  Variable size : Evo -> nat.
  Variable gen : Evo -> Space -> bool.

  Variable evo : Evo.
  Definition n := size evo.

(* The field dst is computed using boolean fields, but in our results
   the propositions are more concise and correspond to the previous papers,
   so we define every field both way and prove the equivalence *)

  Inductive Inp : Time -> Space -> Prop :=
    | Inp_0_c : forall c, gen evo c = true -> Inp 0 c
    | Inp_S_c : forall t c, Inp t c \/ (Inp t (c-1) \/ Inp t (c+1)) -> Inp (S t) c.

  Fixpoint inp t c :=
    match t with
        0 => gen evo c
    | S t => inp t c || (inp t (c-1) || inp t (c+1))
    end.

  Lemma Inp_inp :
    forall t c, Inp t c -> inp t c = true.

    Proof. induction t.
    - intros c H. inversion H. assumption.
    - simpl. intros c H. rewrite -> orb_true_iff. rewrite -> orb_true_iff.
      inversion H. destruct H1 as [H1 | [H1 | H1]].
      * left. apply IHt. assumption.
      * right. left. apply IHt. assumption.
      * right. right. apply IHt. assumption.
    Qed.

  Lemma inp_Inp :
    forall t c, inp t c = true -> Inp t c.
```

```
Proof. induction t.
- simpl. apply Inp_0_c.
- simpl. intros c Hor. apply Inp_S_c. rewrite -> orb_true_iff in Hor.
  rewrite -> orb_true_iff in Hor. destruct Hor as [Hor | [Hor | Hor]].
  * left. apply IHt. assumption.
  * right. left. apply IHt. assumption.
  * right. right. apply IHt. assumption.
Qed.

(* boolean fields are defined before proposition fields because of the if in dst.
   The mutual definition of brd and ins must be split into smaller parts to work *)

 Definition brd_0 t c := (inp t c) && ((c =? 1) || (c =? n)).

 Definition brd_S (brd_l mid_l : Time -> Space -> bool) t c := (brd_l t c) || (mid_l t c).

 Definition ins_0 t c := (inp t c) && ((1 <? c) && (c <? n)).

 Definition ins_S (ins_l sta_l : Time -> Space -> bool) (dst_l : Time -> Space -> nat) t c :=
   match t with
       0 => false
   | S t => (ins_l (S t) c) && ((sta_l (S t) c)
                             &&   ((dst_l (S t) c <? dst_l t (c-1))
                                 ||  (dst_l (S t) c <? dst_l t (c+1))))
   end.

(* dst, sta and mid are defined for every layer l
   There may be a confusion between a function name_l and an argument name_l ? *)

 Fixpoint dst_l (ins_l : Time -> Space -> bool) (t : Time) c :=
   match t with
       0 => 0
   | S t => if (ins_l (S t) c) then 1 + min (dst_l ins_l t (c-1)) (dst_l ins_l t (c+1)) else 0
   end.

 Fixpoint sta_l (brd_l : Time -> Space -> bool) (dst_l : Time -> Space -> nat) t c :=
   match t with
       0 => brd_l 0 c
   | S t => brd_l (S t) c ||    ((dst_l (S t) c =? 1 + dst_l t (c-1))
                              &&         (sta_l brd_l dst_l t (c-1))
                             ||    (dst_l (S t) c =? 1 + dst_l t (c+1))
                              &&         (sta_l brd_l dst_l t (c+1)))
   end.

 Fixpoint mid_l (sta_l : Time -> Space -> bool) (dst_l : Time -> Space -> nat) t c :=
   match t with
       0 => false
   | S t => ((max (dst_l t (c-1)) (dst_l t (c+1)) <? dst_l (S t) c) && (sta_l t (c-1)
                                                                      &&  sta_l t (c+1)))
        ||  ((max (dst_l t (c-1)) (dst_l t (c+1)) =? dst_l (S t) c) && (sta_l t (c-1)
                                                                      && (sta_l t (c+1)
                                                                      &&  sta_l t  c))
```

```
    end.

(* Error: Cannot guess decreasing argument of fix.

  Fixpoint brd l t c :=
    match l with
        0 ⇒ brd_0 t c
    | S l ⇒ brd_S (brd l) (mid l) t c
    end

  with ins l t c :=
    match l with
        0 ⇒ ins_0 t c
    | S l ⇒ ins_S (ins l) (sta l) (dst l) t c
    end

  with dst l t c := dst_l (ins l) t c

  with sta l t c := sta_l (brd l) (dst l) t c

  with mid l t c := mid_l (sta l) (dst l) t c
  .

  So substitutions must be made in the fixpoint to define only brd and ins,
  and the fields dst, sta and mid are defined after. *)

  Fixpoint brd l t c :=
    match l with
        0 ⇒ brd_0 t c
    | S l ⇒ brd_S (brd l) (mid_l (sta_l (brd l) (dst_l (ins l))) (dst_l (ins l))) t c
    end

  with ins l t c :=
    match l with
        0 ⇒ ins_0 t c
    | S l ⇒ ins_S (ins l) (sta_l (brd l) (dst_l (ins l))) (dst_l (ins l)) t c
    end
  .

  Definition dst (l : Level) t c := dst_l (ins l) t c.

  Definition sta l t c := sta_l (brd l) (dst l) t c.

  Definition mid l t c := mid_l (sta l) (dst l) t c.

(* Mutual definition of the proposition fields *)

  Inductive Brd : Level → Time → Space → Prop :=
    | Brd_0_t_c : forall t c, (Inp t c ∧ (c = 1 ∨ c = n)) → Brd 0 t c
    | Brd_S_t_c : forall l t c, (Brd l t c ∨ Mid l t c) → Brd (S l) t c

  with Ins : Level → Time → Space → Prop :=
    | Ins_0_t_c : forall t c, (Inp t c ∧ (1 < c ∧ c < n)) → Ins 0 t c
```

```
  | Ins_S_0_c : forall l c, False → Ins (S l) 0 c
  | Ins_S_S_c : forall l t c, (Ins l (S t) c ∧ (Sta l (S t) c ∧ (
                   dst l (S t) c < dst l t (c−1) ∨ dst l (S t) c < dst l t (c+1)
                   ))) → Ins (S l) (S t) c

with Sta : Level → Time → Space → Prop :=
  | Sta_l_0_c : forall l c, Brd l 0 c → Sta l 0 c
  | Sta_l_S_c : forall l t c, (Brd l (S t) c
                            ∨ ((dst l (S t) c = 1 + dst l t (c−1) ∧ Sta l t (c−1))
                            ∨  (dst l (S t) c = 1 + dst l t (c+1) ∧ Sta l t (c+1)))
    ) → Sta l (S t) c

with Mid : Level → Time → Space → Prop :=
  | Mid_l_0_c : forall l c, False → Mid l 0 c
  | Mid_l_S_c : forall l t c,
        (max (dst l t (c−1)) (dst l t (c+1)) < dst l (S t) c ∧ Sta l t (c−1) ∧ Sta l t (c+1))
      ∨ (max (dst l t (c−1)) (dst l t (c+1)) = dst l (S t) c ∧ Sta l t (c−1) ∧ Sta l t (c+1)
                                                              ∧ Sta l t c)
      → Mid l (S t) c
.

(* Fields: Prop → bool *)

Lemma Brd_brd__Ins_ins__Sta_sta :
  forall l, (forall t c, Brd l t c → brd l t c = true)
         → (forall t c, Ins l t c → ins l t c = true)
         → (forall t c, Sta l t c → sta l t c = true).

  Proof. intros l HB HI. induction t.
  − intros c H. inversion H. apply HB. assumption.
  − intros c H. inversion H. apply orb_true_iff. destruct H2.
    * left. apply HB. assumption.
    * right. apply orb_true_iff. destruct H2.
      + left. destruct H2 as [Hd HS]. apply andb_true_iff. split.
        −− apply Nat.eqb_eq. assumption.
        −− apply IHt. assumption.
      + right. destruct H2 as [Hd HS]. apply andb_true_iff. split.
        −− apply Nat.eqb_eq. assumption.
        −− apply IHt. assumption.
  Qed.

Lemma Brd_brd__Ins_ins__Mid_mid :
  forall l, (forall t c, Brd l t c → brd l t c = true)
         → (forall t c, Ins l t c → ins l t c = true)
         → (forall t c, Mid l t c → mid l t c = true).

  Proof. intros l HB HI. induction t.
  − intros c H. inversion H. contradiction.
  − intros c H. inversion H. apply orb_true_iff. destruct H2.
    * left. destruct H2 as [Hd HS]. apply andb_true_iff. split.
      + apply Nat.leb_le. assumption.
      + destruct HS as [HS1 HS2]. apply andb_true_iff. split.
        −− apply Brd_brd__Ins_ins__Sta_sta. assumption. assumption. assumption.
```

```
      -- apply Brd_brd__Ins_ins__Sta_sta. assumption. assumption. assumption.
  * right. destruct H2 as [Hd HS]. apply andb_true_iff. split.
    + apply Nat.eqb_eq. assumption.
    + destruct HS as [HS1 HS]. apply andb_true_iff. split.
      -- apply Brd_brd__Ins_ins__Sta_sta. assumption. assumption. assumption.
      -- destruct HS as [HS2 HS3]. apply andb_true_iff. split.
         ** apply Brd_brd__Ins_ins__Sta_sta. assumption. assumption. assumption.
         ** apply Brd_brd__Ins_ins__Sta_sta. assumption. assumption. assumption.
Qed.

(* l = 0 *)

 Lemma Brd0_brd0 :
  forall t c, Brd 0 t c → brd 0 t c = true.

  Proof. intros t c HB. inversion HB. destruct H. apply andb_true_iff. split.
  - apply Inp_inp. assumption.
  - apply orb_true_iff. destruct H0.
    * left. apply Nat.eqb_eq. assumption.
    * right. apply Nat.eqb_eq. assumption.
  Qed.

 Lemma Ins0_ins0 :
  forall t c, Ins 0 t c → ins 0 t c = true.

  Proof. intros t c HI. inversion HI. destruct H. apply andb_true_iff. split.
  - apply Inp_inp. assumption.
  - destruct H0. apply andb_true_iff. split.
    * apply Nat.leb_le. assumption.
    * apply Nat.leb_le. assumption.
  Qed.

(* l → S l *)

 Proposition BrdIns_brdins :
  forall l, (forall t c, Brd l t c → brd l t c = true)
        ∧ (forall t c, Ins l t c → ins l t c = true).

  Proof. induction l.
  - split.
    * apply Brd0_brd0.
    * apply Ins0_ins0.
  - destruct IHl as [IHB IHI]. split.
    * intros t c HBS. apply orb_true_iff. inversion HBS. destruct H0.
      + left. apply IHB. assumption.
      + right. apply Brd_brd__Ins_ins__Mid_mid. assumption. assumption. assumption.
    * intros t c HIS. destruct t.
      + inversion HIS. contradiction.
      + inversion HIS. destruct H1 as [HI H1]. apply andb_true_iff. split.
        -- apply IHI. assumption.
        -- destruct H1 as [HS H1]. apply andb_true_iff. split.
           ** assert (Hgoal : sta l (S t) c = true). apply Brd_brd__Ins_ins__Sta_sta.
              apply IHB. apply IHI. apply HS. apply Hgoal.
```

```
          ** apply orb_true_iff. destruct H1.
             ++left. apply Nat.leb_le. assumption.
             ++right. apply Nat.leb_le. assumption.
  Qed.

(* Fields: bool → Prop *)

 Definition Brd_correctness :=
  forall l t c, brd l t c = true ↔ Brd l t c.

 Lemma brd_Brd__ins_Ins__sta_Sta :
  forall l, (forall t c, brd l t c = true → Brd l t c)
        → (forall t c, ins l t c = true → Ins l t c)
        → (forall t c, sta l t c = true → Sta l t c).

  Proof. intros l Hb Hi. induction t.
  - intros c H. apply Sta_l_O_c. apply Hb. apply H.
  - intros c H. apply Sta_l_S_c. apply orb_true_iff in H. destruct H.
    * left. apply Hb. assumption.
    * right. apply orb_true_iff in H. destruct H.
      + left. apply andb_true_iff in H. destruct H as [Hd HS]. split.
        -- apply Nat.eqb_eq in Hd. apply Hd.
        -- apply IHt. apply HS.
      + right. apply andb_true_iff in H. destruct H as [Hd HS]. split.
        -- apply Nat.eqb_eq in Hd. apply Hd.
        -- apply IHt. apply HS.
  Qed.

 Lemma brd_Brd__ins_Ins__mid_Mid :
  forall l, (forall t c, brd l t c = true → Brd l t c)
        → (forall t c, ins l t c = true → Ins l t c)
        → (forall t c, mid l t c = true → Mid l t c).

  Proof. intros l Hb Hi. induction t.
  - intros c H. inversion H.
  - intros c H. apply Mid_l_S_c. apply orb_true_iff in H. destruct H.
    * left. apply andb_true_iff in H. destruct H as [Hd H]. split.
      + apply Nat.leb_le in Hd. apply Hd.
      + apply andb_true_iff in H. destruct H as [Hs1 Hs2]. split.
        -- apply brd_Brd__ins_Ins__sta_Sta. apply Hb. apply Hi. apply Hs1.
        -- apply brd_Brd__ins_Ins__sta_Sta. apply Hb. apply Hi. apply Hs2.
    * right. apply andb_true_iff in H. destruct H as [Hd H]. split.
      + apply Nat.eqb_eq in Hd. apply Hd.
      + apply andb_true_iff in H. destruct H as [Hs1 H]. split.
        -- apply brd_Brd__ins_Ins__sta_Sta. apply Hb. apply Hi. apply Hs1.
        -- apply andb_true_iff in H. destruct H as [Hs2 Hs3]. split.
          ** apply brd_Brd__ins_Ins__sta_Sta. apply Hb. apply Hi. apply Hs2.
          ** apply brd_Brd__ins_Ins__sta_Sta. apply Hb. apply Hi. apply Hs3.
  Qed.

(* l = 0 *)

 Lemma brd0_Brd0 : forall t c, brd 0 t c = true → Brd 0 t c.
```

```
  intros t c Hb. apply andb_true_iff in Hb. destruct Hb. apply Brd_0_t_c. split.
  - apply inp_Inp. assumption.
  - apply orb_true_iff in H0. destruct H0.
    * left. apply Nat.eqb_eq. assumption.
    * right. apply Nat.eqb_eq. assumption.
  Qed.

Lemma ins0_Ins0 : forall t c, ins 0 t c = true -> Ins 0 t c.
  intros t c Hi. apply andb_true_iff in Hi. destruct Hi. apply Ins_0_t_c. split.
  - apply inp_Inp. assumption.
  - apply andb_true_iff in H0. destruct H0. split.
    * apply Nat.leb_le. assumption.
    * apply Nat.leb_le. assumption.
  Qed.

(* l -> S l *)

Proposition brdins_BrdIns :
  forall l, (forall t c, brd l t c = true -> Brd l t c)
        /\ (forall t c, ins l t c = true -> Ins l t c).

  Proof. induction l.
  - split.
    * apply brd0_Brd0.
    * apply ins0_Ins0.
  - destruct IHl as [IHb IHi]. split.
    * intros t c HbS. apply Brd_S_t_c. apply orb_true_iff in HbS. destruct HbS.
      + left. apply IHb. apply H.
      + right. apply brd_Brd__ins_Ins__mid_Mid. apply IHb. apply IHi. apply H.
    * intros t c HiS. destruct t.
      + inversion HiS.
      + apply Ins_S_S_c. apply andb_true_iff in HiS. destruct HiS as [HI HiS]. split.
        -- apply IHi. apply HI.
        -- apply andb_true_iff in HiS. destruct HiS as [HS Hd]. split.
           ** apply brd_Brd__ins_Ins__sta_Sta. apply IHb. apply IHi. apply HS.
           ** apply orb_true_iff in Hd. destruct Hd.
              ++left. apply Nat.leb_le in H. apply H.
              ++right. apply Nat.leb_le in H. apply H.
  Qed.

(* Fields: Prop <-> bool *)

Corollary brd_true_iff :
  forall l t c, Brd l t c <-> brd l t c = true.

  Proof. split.
  - apply BrdIns_brdins.
  - apply brdins_BrdIns.
  Qed.

Corollary ins_true_iff :
  forall l t c, Ins l t c <-> ins l t c = true.
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  Proof. split.
  - apply BrdIns_brdins.
  - apply brdins_BrdIns.
  Qed.

 Corollary sta_true_iff :
  forall l t c, Sta l t c <-> sta l t c = true.

  Proof. split.
  - apply Brd_brd__Ins_ins__Sta_sta. apply brd_true_iff. apply ins_true_iff.
  - apply brd_Brd__ins_Ins__sta_Sta. apply brd_true_iff. apply ins_true_iff.
  Qed.

 Corollary mid_true_iff :
  forall l t c, Mid l t c <-> mid l t c = true.

  Proof. split.
  - apply Brd_brd__Ins_ins__Mid_mid. apply brd_true_iff. apply ins_true_iff.
  - apply brd_Brd__ins_Ins__mid_Mid. apply brd_true_iff. apply ins_true_iff.
  Qed.

(* Beginning of the proof of the monotonicity of the fields *)

 Lemma dst_St :
  forall l t c, dst l (S t) c = if (ins l (S t) c)
                                then 1 + min (dst l t (c-1)) (dst l t (c+1))
                                else 0.

  Proof. reflexivity.
  Qed.

 Lemma dst_local :
  forall l t c, dst l (S t) c <= 1 + min (dst l t (c-1)) (dst l t (c+1)).

  Proof. intros l t c.
  rewrite -> dst_St. destruct (ins l (S t) c).
  - reflexivity.
  - apply le_0_n.
  Qed.

 Lemma ins__dst_incr :
  forall l, (forall t c, Ins l t c -> Ins l (S t) c) -> (forall t c, dst l t c <= dst l (S t) c).

  Proof. intros l HI. induction t.
  - intro c. apply le_0_n.
  - intro c. remember (S t) as St. rewrite -> dst_St. rewrite -> HeqSt.
    rewrite -> HeqSt in IHt. case_eq (ins l (S (S t)) c).
    * intro HiSS. transitivity (S (min (dst l t (c - 1)) (dst l t (c + 1)))).
      apply dst_local. apply le_n_S. apply min_le_min. apply IHt. apply IHt.
    * intro HiSS. case_eq (ins l (S t) c).
      + intro H. apply ins_true_iff in H. apply HI in H. apply ins_true_iff in H.
        apply eq_true_false_abs in H. contradiction. apply HiSS.
      + intro H. rewrite -> dst_St. rewrite -> H. reflexivity.
```

```
  Qed.

Lemma brd__not_ins :
  forall l t c, Brd l t c → (Ins l t c → False).

Proof. induction l.
- intros t c HB HI.
  inversion HI. destruct H as [Hinp1 Hlt]. destruct Hlt as [Hlt1 Hlt2].
  inversion HB. destruct H as [Hinp3 Hor]. destruct Hor.
  * rewrite → H in Hlt1. apply lt_not_le in Hlt1. apply Hlt1. reflexivity.
  * rewrite → H in Hlt2. apply lt_not_le in Hlt2. apply Hlt2. reflexivity.
- intros t c HB HI.
  inversion HI. apply H0. destruct H0 as [HIlS H0]. destruct H0 as [HSlS HdS].
  inversion HB. rewrite ← H1 in H3. destruct H3.
  * apply (IHl (S t0) c).  apply H3. apply HIlS.
  * inversion H3. destruct H8.
    + destruct H8 as [Hd HSl]. apply le_S in Hd. apply le_S_n in Hd.
      apply max_le__and_le in Hd. destruct Hd as [Hd1 Hd2]. destruct HdS.
      -- apply (lt_not_le (dst l (S t0) c) (dst l t0 (c - 1))). apply H8. apply Hd1.
      -- apply (lt_not_le (dst l (S t0) c) (dst l t0 (c + 1))). apply H8. apply Hd2.
    + destruct H8 as [Hdeq HSl].
      assert (max (dst l t0 (c - 1)) (dst l t0 (c + 1)) <= dst l (S t0) c) as Hd.
      rewrite → Hdeq. reflexivity.
      apply max_le__and_le in Hd. destruct Hd as [Hd1 Hd2]. destruct HdS.
      -- apply (lt_not_le (dst l (S t0) c) (dst l t0 (c - 1))). apply H8. apply Hd1.
      -- apply (lt_not_le (dst l (S t0) c) (dst l t0 (c + 1))). apply H8. apply Hd2.
Qed.

Lemma brd_dst0 :
  forall l t c, Brd l t c → dst l t c = 0.

Proof. intros l t c HB.
assert (ins l t c = false) as H.
- apply not_true_is_false. intro HI. apply ins_true_iff in HI.
  apply (brd__not_ins l t c). apply HB. apply HI.
- destruct t.
  * reflexivity.
  * rewrite → dst_St. rewrite → H. reflexivity.
Qed.

Lemma brd__ins__sta_dst :
  forall l, (forall t c, Brd l t c → Brd l (S t) c)
        → (forall t c, Ins l t c → Ins l (S t) c)
        → (forall t c, Sta l t c → dst l t c = dst l (S t) c).

  Proof. intros l HB HI. induction t.
  - intros c HS. inversion HS.
    apply HB in H. apply brd_dst0 in H. rewrite → H. reflexivity.
  - intros c HS. inversion HS. destruct H1.
    * assert (dst l (S t) c = 0) as H3. apply brd_dst0. apply H1. rewrite → H3.
      apply HB in H1. apply brd_dst0 in H1. rewrite → H1.
      reflexivity.
    * destruct H1.
```

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      + destruct H1 as [Hd HSt]. apply IHt in HSt. rewrite → HSt in Hd.
        apply Nat.le_antisymm.
        -- apply ins__dst_incr. apply HI.
        -- rewrite → Hd. transitivity (1 + min (dst l (S t) (c-1)) (dst l (S t) (c+1))).
           apply dst_local. apply le_n_S. apply le_min_l.
      + destruct H1 as [Hd HSt]. apply IHt in HSt. rewrite → HSt in Hd.
        apply Nat.le_antisymm.
        -- apply ins__dst_incr. apply HI.
        -- rewrite → Hd. transitivity (1 + min (dst l (S t) (c-1)) (dst l (S t) (c+1))).
           apply dst_local. apply le_n_S. apply le_min_r.
  Qed.

End Evolution.
```